

Algebra II (Being Reviewed)



Algebra II

CK12 Editor
Kenny Felder

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CHAPTER

1

The Philosophical Introduction No One Reads

Chapter Outline

1.1 [ADVANCED ALGEBRA II COURSE-PACK RALEIGH CHARTER HIGH SCHOOL](#)

1.1 Advanced Algebra II Course-Pack Raleigh Charter High School

The Philosophical Introduction No One Reads

(*but it's real short so please read it anyway)

Welcome to Advanced Algebra II at Raleigh Charter High School! There are three keys to succeeding in this math class.

- a. Do the homework.
- b. Ask questions in class if there is something... anything you don't understand.
- c. Focus on understanding, *doing the problem right*. (Hint: you understand something when you say "Gosh, that makes sense! I should have thought of that myself!")

Here's how it works. The teacher gets up and explains something, and you listen, and it makes sense, and you get it. You work a few problems in class. Then you go home, stare at a problem that looks exactly like the one the teacher put up on the board, and realize you have no idea how to do it. How did that happen? It looked so simple when the teacher did it! Hmm...

So, you dig through your notes, or the book, or you call your friend, or you just try something, and you try something else, and eventually... ta-da! You get the answer! Hooray! Now, you have learned the concept. You didn't learn it in class, you learned it when you figured out how to do it.

Or, let's rewind time a bit. You dig through your notes, you just try something, and eventually... nothing. You still can't get it. That's OK! *that's* why I couldn't get it to work!"

Either way, you win. But if you don't do the homework, then even if the teacher explains the exact same thing in class the next day, it won't help... any more than it helped the previous day.

The materials in this course-pack were originally developed for Mr. Felder's Advanced Algebra II classes in the 2001-2002 school year. Every single student in those classes got an *A* or a *B* on the North Carolina End of Course test at the end of the year. You can too! Do your homework, ask questions in class, and always keep your focus on real understanding. The rest will take care of itself.

CHAPTER **2**

The Function Game

Chapter Outline

2.1 THE FUNCTION GAME

2.1 The Function Game

Each group has three people. Designate one person as the “Leader” and one person as the “Recorder.” (These roles will rotate through all three people.) At any given time, the Leader is looking at a sheet with a list of “functions,” or formulas; the Recorder is looking at the answer sheet. Here’s how it works.

- One of the two players who is not
- The Leader does the formula (silently), comes up with another number, and says it.
- The Recorder writes down both numbers, in parentheses, separated by a comma. (Like a point.)
- Keep doing this until someone guesses the formula. (If someone guesses incorrectly, just keep going.)
- The Recorder now writes down the formula—not in words, but as an algebraic function.
- Then, move on to the next function.

Sound confusing? It’s actually pretty easy. Suppose the first formula was “Add five.” One player says “4” and the Leader says “9”. One player says “-2” and the Leader says “3”. One player says “0” and the Leader says “5”. One player says “You’re adding five” and the Leader says “Correct.” At this point, the Recorder has written down the following:

1. Points: $(4,9)(-2,3)(0,5)$

Answer: $x + 5$

Sometimes there is “-4.” Well, you can’t take the square root of a negative number: -4 is not in your *domain*, meaning the set of numbers you are allowed to work on. So you respond that “ -4 is not in my domain.”

Leader, do not

The Function Game Leader’s Sheet

Only the leader should look at this sheet. Leader, use a separate sheet to cover up all the functions below the one you are doing right now. That way, when the roles rotate, you will only have seen the ones you’ve done.

- Double the number, then add six.
- Add three to the number, then double.
- Multiply the number by -1 , then add three.
- Subtract one from the number. Then, compute one divided by
- Divide the number by two.
- No matter what number you are given, always answer “ -3 .”
- Square the number, then subtract four.
- If you are given a positive number *If you are given a negative number*, multiply that number by -1 .
- Cube the number.
- Add two to the number. Also, subtract two from the original number. Multiply these two answers
- Take the square root of the number. Round up to the nearest integer.
 - Add one to the number, then square.
- Square the number, then add 1.
- Cube the number. Then subtract the original number from that answer.
- Give back the lowest prime number *greater than or equal to* the number.
- If you are given an odd number, respond 1. If you are given an even number, respond 2. (Fractions are not in the domain of this function.)

The Function Game: Answer Sheet Recorder

1. Points:

Answer:

2. Points:

Answer:

3. Points:

Answer:

4. Points:

Answer:

5. Points:

Answer:

6. Points:

Answer:

7. Points:

Answer:

8. Points:

Answer:

9. Points:

Answer:

10. Points:

Answer:

11. Points:

Answer:

12. Points:

Answer:

13. Points:

Answer:

14. Points:

Answer:

15. Points:

Answer:

16. Points:

Answer:

Name: _____

Homework: The Function Game

1. Describe in words what a variable *function* is.

There are seven functions below (numbered 2-8). For each function,

- Write the same function in algebraic notation.
- Generate three points from that function.

For instance, if the function were “Add five” the algebraic notation would be “ $x + 5$ ”. The three points might be $(2, 7)$, $(3, 8)$, and $(-5, 0)$.

2. Triple the number, then subtract six.

a. Algebraic notation: _____

b. Three points: _____

3. Return 4, no matter what.

a. Algebraic notation: _____

b. Three points: _____

4. Add one. Then take the square root of the result. Then, divide that

a. Algebraic notation: _____

b. Three points: _____

5. Add two to the original number. Subtract two from the original number. Then, multiply those two answers together.

a. Algebraic notation: _____

b. Three points: _____

6. Subtract two, then triple.

a. Algebraic notation: _____

b. Three points: _____

7. Square, then subtract four.

a. Algebraic notation: _____

b. Three points: _____

8. Add three. Then, multiply by four. Then, subtract twelve. Then, divide by the original number.

a. Algebraic notation: _____

b. Three points: _____

9. In some of the above cases, two functions always give the same answer, even though they are different functions. We say that these functions are “equal” to each other. For instance, the function “add three and then subtract five” is equal to the function “subtract two” because they always give the same answer. (Try it, if you don’t believe me!) We can write this as:

$$x + 3 - 5 = x - 2$$

Note that this is not an equation you can solve for x —it is a generalization which is true for *all* x values. It is a way of indicating that if you do the calculation on the left, and the calculation on the right, they will always give you the same answer.

In the functions 2 – 8 above, there are three such pairs of “equal” functions. Which ones are they? Write the algebraic equations that state their equalities (like my $x + 3 - 5 = x - 2$ equation).

10. Of the following sets of numbers, there is one that could not possibly have been generated by any function whatsoever. Which set it is, and why? (No credit unless you explain why!)

- $(3, 6)(4, 8)(-2, -4)$
- $(6, 9)(2, 9)(-3, 9)$
- $(1, 112)(2, -4)(3, 3)$
- $(3, 4)(3, 9)(4, 10)$
- $(-2, 4)(-1, 1)(0, 0)(1, 1)(2, 4)$

Name: _____

Homework: Functions in the Real World

Laura is selling doughnuts for 35¢ each. Each customer fills a box with however many doughnuts he wants, and then brings the box to Laura to pay for them. Let n represent the number of doughnuts in a box, and let c represent the cost of the box (in cents).

- If the box has 3 doughnuts, how much does the box cost?
- If $c = 245$, how much does the box cost? How many doughnuts does it have?
- If a box has n doughnuts, how much does it cost?
- Write a function $c(n)$ that gives the *cost of a box*, as a function of the *number of doughnuts in the box*.

2. Worth is doing a scientific study of graffiti in the downstairs boy’s room. On the first day of school, there is no graffiti. On the second day, there are two drawings. On the third day, there are four drawings. He forgets to check on the fourth day, but on the fifth day, there are eight drawings. Let d represent the day, and g represent the number of graffiti marks that day.

- Fill in the following table, showing Worth’s four data points.

d (day)

g (number of graffiti marks)

- If this pattern keeps up, how many graffiti marks will there be on day 10?
- If this pattern keeps up, on what day will there be 40 graffiti marks?
- Write a function $g(d)$ that gives the *number of graffiti marks* as a function of the *day*.

3. Each of the following is a set of points. Next to each one, write “yes” if that set of points could have been generated *could not have been generated* by a function. (You do *not* have to figure out what the function is. But you may want to try for fun—I didn’t just make up numbers randomly. . .)

- $(1, -1)(3, -3)(-1, -1)(-3, -3)$ _____
 - $(1, \pi)(3, \pi)(9, \pi)(\pi, \pi)$ _____
 - $(1, 1)(-1, 1)(2, 4)(-2, 4)(3, 9)(-3, 9)$ _____
 - $(1, 1)(1, -1)(4, 2)(4, -2)(9, 3)(9, -3)$ _____
 - $(1, 1)(2, 3)(3, 6)(4, 10)$ _____
4. $f(x) = x^2 + 2x + 1$.
- $f(2) =$

b. $f(-1) =$

c. $f\left(\frac{3}{2}\right) =$

d. $f(y) =$

e. $f(\text{spaghetti}) =$

f. $f(\sqrt{x}) =$

g. $f(f(x)) =$

5. Make up a function that has something to do with movies.

a. Think of a scenario where there are two numbers, one of which depends on the other. Describe the scenario, clearly identifying the independent variable *dependent variable*.

b. Write the function that shows how the dependent variable depends on the independent variable.

c. Now, plug in an example number to show how it works.

Algebraic Generalizations

1. a. Pick a number: _____

b. Add three: _____

c. Subtract three from your answer in part (b): _____

d. What happened? _____

e. Write an algebraic generalization to represent this rule. _____

f. Is there any number for which this rule will not

2. a. Pick a number: _____

b. Subtract five: _____

c. Double your answer in part (b): _____

d. Add ten to your answer in part (c): _____

e. Divide your answer in part (d) by your original number (a): _____

f. Now, repeat that process for three different numbers. Record the number you started with (a) and the number you ended up with (e).

Started with: _____

Ended with: _____

Started with: _____

Ended with: _____

Started with: _____

Ended with: _____

g. What happened? _____

h. Write an algebraic generalization to represent this rule. _____

i. Is there any number for which this rule will not

3. Here are the first six powers of two.

- $2^1 = 2$

- $2^2 = 4$

- $2^3 = 8$
- $2^4 = 16$
- $2^5 = 32$
- $2^6 = 64$

- a. If I asked you for 2^7 (without a calculator), how would you get it? More generally, how do you always get from one term in this list to the next term? _____
- b. Write an algebraic generalization to represent this rule. _____
4. Now, we're going to make that rule even more general. Suppose I want to multiply 2^5 times 2^3 . Well, 2^5 means $2*2*2*2*2$, and 2^3 means $2*2*2$. So we can write the whole thing out like this.

$$\underbrace{2^5}_{2*2*2*2*2} * \underbrace{2^3}_{2*2*2} = \underbrace{2^8}_{2*2*2*2*2*2*2*2}$$

This shows that $(2^5)(2^3) = 2^8$.

- a. Using a similar drawing, demonstrate what $(10^3)(10^4)$ must be.
- b. Now, write an algebraic generalization for this rule. _____
5. The following statements are true.

- $3 \times 4 = 4 \times 3$
- $7 \times -3 = -3 \times 7$
- $\frac{1}{2} \times 8 = 8 \times \frac{1}{2}$

Write an algebraic generalization for this rule. _____

6. Look at the following pairs

$8 \times 8 = 64$

$5 \times 5 = 25$

$10 \times 10 = 100$

$3 \times 3 = 9$

$7 \times 9 = 63$

$4 \times 6 = 24$

$9 \times 11 = 99$

$2 \times 4 = 8$

- a. Based on these pairs, if I told you that $30 \times 30 = 900$, could you tell me (immediately, without a calculator) what 29×31 is? _____
- b. Express this rule—the pattern in these numbers—in words
- c. Whew! That was complicated, wasn't it? Good thing we have math. Write the algebraic generalization for this rule. _____
- d. Try out this generalization with negative numbers, zero, and with *fractions*. (Show your work below, trying all three of these cases separately.) Does it *always* work, or are there cases where it doesn't?

Name: _____

Homework: Algebraic Generalizations

In class, we talked about the following four pairs of statements.

$8 \times 8 = 64$

$5 \times 5 = 25$

$10 \times 10 = 100$

$3 \times 3 = 9$

$7 \times 9 = 63$

$4 \times 6 = 24$

$9 \times 11 = 99$

$2 \times 4 = 8$

You made an algebraic generalization about these statements: write that generalization again below.

Now, we are going to generalize it further. Let's focus on the 10×10 thing.

$$10 \times 10 = 100$$

There are two numbers that are one away 10; these numbers are, of course, 9 and 11. As we saw, 9×11 is 99. It is *one less* than 100.

Now, suppose we look at the two numbers that are two away 10? Or *three away*? Or *four away*? We get a sequence like this (fill in all the missing numbers):

| | |
|--|--|
| $10 \times 10 = 100$ | |
| $9 \times 11 = 99$ | 1 away from 10, the product is 1 less than 100 |
| $8 \times 12 = \underline{\quad}$ | 2 away from 10, the product is <u> </u> less than 100 |
| $7 \times 13 = \underline{\quad}$ | 3 away from 10, the product is <u> </u> less than 100 |
| $\underline{\quad} \times \underline{\quad} = \underline{\quad}$ | <u> </u> away from 10, the product is <u> </u> less than 100 |
| $\underline{\quad} \times \underline{\quad} = \underline{\quad}$ | <u> </u> away from 10, the product is <u> </u> less than 100 |

Do you see the pattern? What would you expect to be the next sentence in this sequence?

Write the algebraic generalization for this rule.

Does that generalization work when the “ away from 10” is 0? Is a fraction? Is a negative number? Test all three cases. (Show your work!)

Name: _____

Homework: Graphing

The following graph shows the temperature throughout the month of March. Actually, I just made this graph up—the numbers do not actually reflect the temperature throughout the month of March. We're just pretending, OK?

1. Give a weather report for the month of March, in words.
2. On what days was the temperature exactly 0°C ?
3. On what days was the temperature below freezing?
4. On what days was the temperature above freezing?
5. What is the domain of this graph?
6. During what time periods was the temperature going up?
7. During what time periods was the temperature going down?
8. Mary started a company selling French Fries over the Internet. For the first 3 days, while she worked on the technology, she lost \$100 per day. Then she opened for business. People went wild over her French Fries! She made \$200 in one day, \$300 the day after that, and \$400 the day after that. The following day she was sued by an angry customer who discovered that Mary had been using genetically engineered potatoes. She lost \$500 in the lawsuit that day and closed up her business. Draw a graph showing Mary's profits as a function of days.
9. Fill in the following table. Then, based on your table, draw graphs of the functions $y = x^2$, $y = x^2 + 2$, $y = x^2 - 1$, $y = (x + 3)^2$, $y = 2x^2$, and $y = -x^2$.

TABLE 2.1:

| | | | | | | |
|-----|-------|-----------|-----------|-------------|--------|--------|
| x | x^2 | $x^2 + 2$ | $x^2 - 1$ | $(x + 3)^2$ | $2x^2$ | $-x^2$ |
| -3 | | | | | | |

TABLE 2.1: (continued)

| x | x^2 | $x^2 + 2$ | $x^2 - 1$ | $(x + 3)^2$ | $2x^2$ | $-x^2$ |
|-----|-------|-----------|-----------|-------------|--------|--------|
| -2 | | | | | | |
| -1 | | | | | | |
| 0 | | | | | | |
| 1 | | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |

Now describe in words what happened. . .

- How did adding 2 to the function change the graph?
- How did subtracting 1 from the function change the graph?
- How did adding three to x change the graph?
- How did doubling the function change the graph?
- How did multiplying the graph by -1 change the graph?
- By looking at your graphs, estimate the point of intersection of the graphs $y = x^2$ and $y = (x + 3)^2$. What does this point represent?

Name: _____

Horizontal and Vertical Permutations

1. Standing at the edge of the Bottomless Pit of Despair, you kick a rock off the ledge and it falls into the pit. The height of the rock is given by the function $h(t) = -16t^2$, where t is the time since you dropped the rock, and h is the height of the rock.

- Fill in the following table.

| | | | | | | | | |
|----------------|---|---------------|---|----------------|---|----------------|---|----------------|
| time (seconds) | 0 | $\frac{1}{2}$ | 1 | $1\frac{1}{2}$ | 2 | $2\frac{1}{2}$ | 3 | $3\frac{1}{2}$ |
| height (feet) | | | | | | | | |

- $h(0) = 0$. What does that tell us about the rock?
- All the other heights are negative: what does that tell us about the rock?
- Graph the function $h(t)$. Be sure to carefully label your axes!

2. Another rock was dropped at the exact same time as the first rock; but instead of being kicked from the ground, it was dropped from your hand, 3 feet up. So, as they fall, the second rock is always three feet higher than the first rock.

- Fill in the following table for the second

| | | | | | | | | |
|----------------|---|---------------|---|----------------|---|----------------|---|----------------|
| time (seconds) | 0 | $\frac{1}{2}$ | 1 | $1\frac{1}{2}$ | 2 | $2\frac{1}{2}$ | 3 | $3\frac{1}{2}$ |
| height (feet) | | | | | | | | |

- Graph the function $h(t)$ for the new rock. Be sure to carefully label your axes!
- How does this new function $h(t)$ compare to the old one? That is, if you put them side by side, what change would you see?

- d. The original function was $h(t) = -16t^2$. What is the new function? $h(t) =$
(*make sure the function you write actually generates the points in your table!)
- e. Does this represent a horizontal permutation *vertical permutation*?
- f. Write a generalization based on this example, of the form: when you do such-and-such *such-and-such* a way.
3. A third rock was dropped from the exact same place as the first rock (kicked off the ledge, but it was dropped $h = 0$) at that time.
- a. Fill in the following table for the third

| | | | | | | | | | | | |
|----------------|---|---------------|---|----------------|---|----------------|---|----------------|---|----------------|---|
| time (seconds) | 0 | $\frac{1}{2}$ | 1 | $1\frac{1}{2}$ | 2 | $2\frac{1}{2}$ | 3 | $3\frac{1}{2}$ | 4 | $4\frac{1}{2}$ | 5 |
| height (feet) | 0 | 0 | 0 | 0 | | | | | | | |

- b. Graph the function $h(t)$ for the new rock. Be sure to carefully label your axes!
- c. How does this new function $h(t)$ compare to the *original* one? That is, if you put them side by side, what change would you see?
- d. The original function was $h(t) = -16t^2$. What is the new function? $h(t) =$
(*make sure the function you write actually generates the points in your table!)
- e. Does this represent a horizontal permutation *vertical permutation*?
- f. Write a generalization based on this example, of the form: when you do such-and-such *such-and-such* a way.
- Name: _____

Homework: Horizontal and Vertical Permutations

1. In a certain magical bank, your money doubles every year. So if you start with \$1, your money is represented by the function $M = 2^t$, where t is the time (in years) your money has been in the bank, and M is the amount of money (in dollars) you have.

Don puts \$1 into the bank at the very beginning ($t = 0$).

Susan also \$1 into the bank when $t = 0$. However, she also has a secret stash of \$2 under her mattress at home. Of course, her \$2 stash doesn't grow: so at any given time t , she has the same amount of money that Don has, plus \$2 more.

Cheryl, like Don, starts with \$1. But during the first year, she hides it under *her* mattress. After a year ($t = 1$) she puts it into the bank, where it starts to accrue interest.

- a. Fill in the following table to show how much money each person has.

TABLE 2.2:

| | $t = 0$ | $t = 1$ | $t = 2$ | $t = 3$ |
|---------------|---------|---------|---------|---------|
| Don | 1 | | | |
| Susan | 3 | | | |
| Cheryl | 1 | 1 | | |

- b. Graph each person's money as a function of time.
- c. Below each graph, write the function that gives this person's money as a function of time. Be sure your function correctly generates the points you gave above! (*For Cheryl, your function will not accurately represent her money between $t = 0$ and $t = 1$, but it should accurately represent it thereafter.)

2. The function $y = f(x)$ is defined on the domain $[-4, 4]$ as shown below.

- What is $f(-2)$? (That is, what does this function give you if you give it a -2?)
- What is $f(0)$?
- What is $f(3)$?
- The function has three zeros. What are they?

The function $g(x)$ is defined by the equation: $g(x) = f(x) - 1$. That is to say, for any x -value you put into $g(x)$, it first puts that value into $f(x)$, and then it subtracts 1 from the answer.

- What is $g(-2)$?
- What is $g(0)$?
- What is $g(3)$?
- Draw $y = g(x)$ next to the $f(x)$ drawing above.

The function $h(x)$ is defined by the equation: $h(x) = f(x + 1)$. That is to say, for any x -value you put into $h(x)$, it first adds 1 to that value, and then it puts the new x -value into $f(x)$.

- What is $h(-3)$?
- What is $h(-1)$?
- What is $h(2)$?
- Draw $y = h(x)$ next to the $f(x)$ drawing to the right.

m. Which of the two permutations above changed the domain

3. On your calculator, graph the function $Y1 = x^3 - 13x - 12$. Graph it in a window with x going from -5 to 5 , and y going from -30 to 30 .

- Copy the graph below. Note the three zeros at $x = -3, x = -1$, and $x = 4$.
- For what x -values is the function *less than zero*? (Or, to put it another way: solve the inequality $x^3 - 13x - 12 < 0$.)
- Construct a function that looks exactly like this function, but moved up $Y2$, so you can see the two functions together). When you have a function that works, write your new function below.
- Construct a function that looks exactly like the original function, but moved
- Construct a function that looks exactly like the original function, but moved down 1 unit *to the right*. When you have a function that works, write your new function below.

Name: _____

Sample Test: Functions I

1. Chris is $1\frac{1}{2}$ years younger than his brother David. Let D represent David's age, and C represent Chris's age.

- If Chris is fifteen years old, how old is David? _____
- Write a function to show how to find David's age, given Chris's age. $D(C) =$ _____

2. Sally slips into a broom closet, waves her magic wand, and emerges as... the candy bar fairy! Flying through the window of the classroom, she gives every student two candy bars. Then five

Let s represent the number of students in the class, and c represent the total number of candy bars distributed. Two for each student, and five for the teacher.

- Write a function to show how many candy bars Sally gave out, as a function of the number of students. $c(s) =$ __-

- Use that function to answer the question: if there were 20 students in the classroom, how many candy bars were

distributed? *First represent the question in functional notation—then answer it.* _____

c. Now use the same function to answer the question: if Sally distributed 35 candy bars, how many students were in the class? *First represent the question in functional notation—then answer it.* _____

3. The function $f(x)$ is “Subtract three, then take the square root.”

a. Express this function algebraically, instead of in words: $f(x) =$ _____

b. Give any three points that could be generated by this function: _____

c. What x -values are in the domain of this function? _____

4. The function $y(x)$ is “Given any number, return 6.”

a. Express this function algebraically, instead of in words: $y(x) =$ _____

b. Give any three points that could be generated by this function: _____

c. What x -values are in the domain of this function? _____

5. $z(x) = x^2 - 6x + 9$

a. $z(-1) =$ _____

b. $z(0) =$ _____

c. $z(1) =$ _____

d. $z(3) =$ _____

e. $z(x+2) =$ _____

f. $z(z(x)) =$ _____

6. Of the following sets of numbers, indicate which ones could possibly have been generated by a function. All I need is a “Yes” or “No”—you don’t have to tell me the function! (But, you may do so if you would like to...)

a. $(-2, 4)(-1, 1)(0, 0)(1, 1)(2, 4)$ Yes No

b. $(4, -2)(1, -1)(0, 0)(1, 1)(4, 2)$ Yes No

c. $(2, \pi)(3, \pi)(4, \pi)(5, 1)$ Yes No

d. $(\pi, 2)(\pi, 3)(\pi, 4)(1, 5)$ Yes No

7. Make up a function involving music

a. Write the scenario. Your description should clearly tell me—in words—how one value depends on another.

b. Name, and clearly describe, two variables. Indicate which is dependent/independent.

c. Write a function showing how the dependent variable depends on the independent variable. If you were explicit enough in parts (a) and (b), I should be able to predict your answer to part (c) before I read it.

d. Choose a sample number to show how your function works. Explain what the result means.

8. Here is an algebraic generalization: for any number x , $x^2 - 25 = (x + 5)(x - 5)$.

a. Plug $x = 3$ into that generalization, and see if it works.

b. 20×20 is 400. Given that, and the generalization, can you find 15×25 without a calculator? (Don’t just give me the answer, show how you got it!)

9. Amy has started a company selling candy bars. Each day, she buys candy bars from the corner store and sells them to students during lunch. The following graph shows her profit

a. On what days did she break even?

b. On what days did she lose

10. The picture to the right shows the graph of $y = \sqrt{x}$. The graph starts at $(0,0)$ and moves up and to the right forever.

- What is the domain of this graph?
- Write a function that looks exactly the same, except that it starts at the point $(-3, 1)$ and moves up-and-right from there.

11. The following graph represents the graph $y = f(x)$.

- Is it a function? Why or why not?
- What are the zeros?
- For what x -values is it positive?
- For what x -values is it negative?
- To the right is the same function $f(x)$. On that same graph, draw the graph of $y = f(x) - 2$.
- To the right is the same function $f(x)$. On that same graph, draw the graph of $y = -f(x)$.

Extra credit: *below* it is easy to square.

Suppose I want to find 31^2 . That's hard. But it's easy to find 30^2 , that's 900. Now, here comes the trick: add 30, and then add 31. $900 + 30 + 31 = 961$. That's the answer! $31^2 = 961$.

- Use this trick to find 41^2 . (Don't just show me the answer, show me the work!)
- Write the algebraic generalization that represents this trick.

Name: _____

Lines

- You have \$150 at the beginning of the year. (Call that day "0.") Every day you make \$3.
 - How much money do you have on day 1?
 - How much money do you have on day 4?
 - How much money do you have on day 10?
 - How much money do you have on day n ? This gives you a general function for how much money you have on any given day.
 - How much is that function going up every day? This is the slope of the line.
 - Graph the line.
- Your parachute opens when you are 2,000 feet above the ground. (Call this time $t = 0$.) Thereafter, you fall 30 feet every second. (Note: I don't know anything about skydiving, so these numbers are probably not realistic!)
 - How high are you after one second?
 - How high are you after ten seconds?
 - How high are you after fifty seconds?
 - How high are you after t seconds? This gives you a general formula for your height.
 - How long does it take you to hit the ground?
 - How much altitude are you gaining every second? This is the slope of the line. Because you are falling, you are actually gaining negative

g. Graph the line.

3. Make up a word problem like numbers 1 and 2. Be very clear about the independent and dependent variables, as always. Make sure the relationship between them is linear! Give the general equation and the slope of the line.

4. Compute the slope of a line that goes from $(1, 3)$ to $(6, 18)$.

5. For each of the following diagrams, indicate roughly what the slope is.

a.

b.

c.

d.

e.

f.

6. Now, for each of the following graphs, draw a line with roughly the slope indicated. For instance, on the first little graph, draw a line with slope 2.

a.

b.

c.

For problems 7 and 8,

- Solve for y , and put the equation in the form $y = mx + b$ (. . . if it isn't already in that form)
- Identify the slope
- Identify the y - intercept, and graph it
- Use the slope to find one point other than the y - intercept on the line
- Graph the line

7. $y = 3x - 2$

Slope: _____

y - intercept: _____

Other point: _____

8. $2y - x = 4$

Equation in $y = mx + b$ form:

Slope: _____

y - intercept: _____

Other point: _____

Name: _____

Homework: Graphing Lines

1. $2y + 7x + 3 = 0$ is the equation for a line.

a. Put this equation into the "slope-intercept" form $y = mx + b$

b. slope = _____

c. y - intercept = _____

d. x - intercept = _____

e. Graph it

2. The points $(5, 2)$ and $(7, 8)$ lie on a line.

a. Find the slope of this line

b. Find another point on this line

3. When you're building a roof, you often talk about the "pitch" of the roof—which is a fancy word that means its slope. You are building a roof shaped like the following. The roof is perfectly symmetrical. The slope of the left-hand side is $\frac{1}{3}$. In the drawing below, the roof is the two thick black lines—the ceiling of the house is the dotted line 60' long.

a. What is the slope of the right-hand side of the roof?

b. How high is the roof? That is, what is the distance from the ceiling of the house, straight up to the point at the top of the roof?

c. How long is the roof? That is, what is the combined length of the two thick black lines in the drawing above?

4. In the equation $y = 3x$, explain why 3 is the slope. (Don't just say, "because it's the m in $y = mx + b$." Explain why $\frac{\Delta y}{\Delta x}$ will be 3 for any two points on this line, just like we explained in class why b is the y -intercept.)

5. How do you measure the height of a very tall mountain? You can't just sink a ruler down from the top to the bottom of the mountain!.

So here's one way you could do it. You stand behind a tree, and you move back until you can look straight over the top of the tree to the top of the mountain. Then you measure the height of the tree, the distance from you to the mountain, and the distance from you to the tree. So you might get results like this.

How high is the mountain?

6. The following table shows how much money Scrooge McDuck has been worth every year since 1999.

| | | | | | | |
|-----------|--------------|----------------|--------------|----------------|--------------|-----------------|
| Year | 1999 | 2000 | 2001 | 2002 | 2003 | 2004 |
| Net worth | \$3 Trillion | \$4.5 Trillion | \$6 Trillion | \$7.5 Trillion | \$9 Trillion | \$10.5 Trillion |

a. How much is a trillion, anyway?

b. Graph this relation.

c. What is the slope of the graph?

d. How much money can Mr. McDuck earn in 20 years at this rate?

7. Make up and solve your own word problem using slope.

Composite Functions

1. You are the foreman at the Sesame Street Number Factory. A huge conveyer belt rolls along, covered with big plastic numbers for our customers. Your two best employees are Katie and Nicolas. Both of them stand at their stations by the conveyer belt. Nicolas's job is: whatever number comes to your station, add *subtract* 10, and send the result down the line to Sesame Street.

a. Fill in the following table.

This number comes the line -5 -3 -1 2 4 6 10 x $2x$
 Nicolas comes up with this number, and
 sends it down the line to Katie
 Katie then spits out this number

b. In a massive downsizing effort, you are going to fire Nicolas. Katie is going to take over *both* functions (Nicolas's and her own). Katie is given a number; first she takes care of Nicolas's function, and then her own. But now Katie is overworked, so she comes up with a shortcut: one function she can do, that covers both Nicolas's job and her own. What does Katie do to each number you give her? (Answer in words.)

2. Taylor is driving a motorcycle across the country. Each day he covers 500 miles. A policeman started the same place Taylor did, waited a while, and then took off, hoping to catch some illegal activity. The policeman stops each day exactly five miles behind Taylor.

Let d equal the number of days they have been driving. (So after the first day, $d = 1$.) Let T be the number of miles Taylor has driven. Let p equal the number of miles the policeman has driven.

- After three days, how far has Taylor gone? _____
- How far has the policeman gone? _____
- Write a function $T(d)$ that gives the *number of miles Taylor has traveled*, as a function of how many days he has been traveling. _____
- Write a function $p(T)$ that gives the *number of miles the policeman has traveled*, as a function of the distance that Taylor has traveled. _____
- Now write the composite function $p(T(d))$ that gives the *number of miles the policeman has traveled*, as a function of the number of days he has been traveling.

3. Rashmi is an honor student by day; but by night, she works for the university as a tutor. Each month she gets paid a \$1000 base salary, *plus* an extra \$100 for each person she tutors. The university pays all tutors in \$20 bills.

Let k equal the number of people Rashmi kills tutors in a given month. Let m be the amount of money she is paid that month, in dollars. Let b be the number of \$20 bills she gets.

- Write a function $m(k)$ that tells how much money Rashmi makes, in a given month, as a function of the number of people she tutors. _____
- Write a function $b(m)$ that tells how many bills Rashmi gets, in a given month, as a function of the number of dollars she makes. _____
- Write a composite function $b(m(k))$ that gives the number of bills Rashmi gets, as a function of the number of people she tutors.
- If Rashmi kills 5 students in a month, how many \$20 bills does she earn? *First, translate this question into function notation—then solve it for a number.*
- If Rashmi earns 100 \$20 bills in a month, how many students did she tutor? *First, translate this question into function notation—then solve it for a number.*

4. Make up a problem like numbers 2 and 3. Be sure to take all the right steps: *define the scenario, define your variables clearly*, and then show the functions that relate the variables. This is just like the problems we did last week, except that you have to use three variables, related by a composite function.

5. $f(x) = \sqrt{x} + 2$. $g(x) = x^2 + x$.

a. $f(7) =$ _____

b. $g(7) =$ _____

e. $g(x) = (x + 2)$

f. $g(x) = \frac{x}{x^2 + 3x + 4}$

5. $h(x) = 4x$. $h(i(x)) = x$. Can you find what function $i(x)$ is, to make this happen?

Name: _____

Inverse Functions

1. We are playing the function game. Every time you give Christian a number, he doubles it and subtracts six.

a. If you give Christian a ten, what will he give you back?

b. If you give Christian an x , what will he give you back?

c. What number would you give Christian, that would make him give you a 0?

d. What number would you give Christian, that would make him give you a ten?

e. What number would you give Christian, that would make him give you an x ? (*Hint for the stuck*: try to follow the process you used to answer part d.)

2. A television set dropped from the top of a 300' building falls according to the equation:

$$h(t) = 300 - 16t^2$$

where t is the amount of time that has passed since it was dropped (measured in seconds), and h is the height of the television set above ground (measured in feet).

a. Where is the television set after 0 seconds have elapsed?

b. Where is the television set after 2 seconds have elapsed?

c. A man is watching out of the window of the first floor, 20' above ground. At what time does the television set go flying by?

d. At what time does the television reach the ground?

e. Find a general formula $t(h)$ that can be used to quickly and easily answer all questions like (c) and (d).

Find the inverse of each function. For each one,

3. $y = x + 5$

• Inverse function:

• Test:

• Test:

4. $y = x - 6$

• Inverse function:

• Test:

- Test:

5. $y = 3x$

- Inverse function:

- Test:

- Test:

6. $y = \frac{x}{4}$

- Inverse function:

- Test:

- Test:

7. $y = 3x + 12$

- Inverse function:

- Test:

- Test:

8. $y = \frac{100}{x}$

- Inverse function:

- Test:

- Test:

9. $y = \frac{2x+3}{7}$

- Inverse function:

- Test:

- Test:

10. $y = x^2$

- Inverse function:

- Test:

- Test:

11. $y = 2^x$

- Inverse function:

- Test:

- Test:

Name: _____

Homework: Inverse Functions

1. On our last “Sample Test,” we did a scenario where Sally distributed two candy bars to each student and five to the teacher. We found a function $c(s)$ that represented how many candy bars she distributed, as a function of the number of students in the room.

- What was that function again?
- How many candy bars would Sally distribute if there were 20 students in the room?
- Find the inverse function.
- Now—this is the key part—explain what that inverse function actually represents. Ask a word-problem question that I can answer by using the inverse function.

2. Make up a problem like 1. That is, make up a scenario, and show the function that represents that scenario. Then, give a word problem that is answered by the *inverse* function, and show the inverse function.

For each function, find the inverse function, the domain, and the range

3. $y = 2 + \frac{1}{x}$

4. $\frac{2x+3}{7}$

5. $2(x+3)$

6. x^2

7. x^3

8. \sqrt{x}

9. $y = \frac{2x+1}{x}$

10. $y = \frac{x}{2x+1}$

11. “The functions $f(x)$ and $g(x)$ are inverse functions.” Express that sentence in math, instead of in words (or using as few words as possible).

TAPPS Exercise: How Do I Solve That For y?

OK, so you’re looking for the inverse function of $y = \frac{x}{2x+1}$. So you reverse the x and the y and you come up with $x = \frac{y}{2y+1}$. Now you have to solve that for y , and you’re stuck.

First of all, let’s review what that means! To “solve it for $y = something$,” where the *something* has no y in it anywhere. So $y = 2x + 4$ is solved for y , but $y = 2x + 3y$ is not. Why? Because in the first case, if I give you x , you can immediately find y . But in the second case, you cannot.

“Solving it for y ” is also sometimes called “isolating y ” because you are getting y all alone.

So that’s our goal. How do we accomplish it?

$$x = \frac{y}{2y + 1}$$

1. The biggest problem we have is the fraction. To get rid of it, we multiply both sides by $2y + 1$.

$$x(2y + 1) = y$$

2. Now, we distribute through.

$$2xy + x = y$$

3. Remember that our goal is to isolate y . So now we get all the things *with* y on one side, and all the things *without* y on the other side.

$$x = y - 2xy$$

4. Now comes the key step: we factor out y from all the terms on the right side. This is the distributive property (like we did in step 2) done in reverse, and you should check it by distributing through.

$$x = y(1 - 2x)$$

5. Finally, we divide both sides by what is left in the parentheses!

$$\frac{x}{1 - 2x} = y$$

Ta-da! We're done! $\frac{x}{1-2x}$ is the inverse function of $\frac{x}{2x+1}$. Not convinced? Try two tests.

Test 1:

Test 2:

Now, you try it! Follow the above steps one at a time. You should switch roles at this point: the previous student *teacher*. Your job: find the inverse function of $y = \frac{x+1}{x-1}$.

Name: _____

Sample Test: Functions II

1. Joe and Lisa are baking cookies. Every cookie is a perfect circle. Lisa is experimenting with cookies of different radii (*the plural of "radius"). Unknown to Lisa, Joe is very competitive about his baking. He sneaks in to measure the radius of Lisa's cookies, and then makes his own 2" bigger radius.

Let L be the radius of Lisa's cookies. Let J be the radius of Joe's cookies. Let a be the *area* of Joe's cookies.

a. Write a function $J(L)$ that shows the radius of Joe's cookies as a function of the radius of Lisa's cookies.

b. Write a function $a(J)$ that shows the area of Joe's cookies as a function of their radius. (If you don't know the area of a circle, ask me—this information will cost you 1 point.)

c. Now, put them together into the function $a(J(L))$ that gives the *area* of Joe's cookies, as a direct function of the *radius* of Lisa's.

- d. Using that function, answer the question: if Lisa settles on a 3" radius, what will the area of Joe's cookies be? First, write the question in function notation—then solve it
- e. Using the same function, answer the question: if Joe's cookies end up 49π square inches in area, what was the radius of Lisa's cookies? *First, write the question in function notation—then solve it.*
2. Make up a word problem involving composite functions, *superheroes*.
- a. Describe the scenario. Remember that it must have something *something else* that depends on *still another thing*. If you have described the scenario carefully, I should be able to guess what your variables will be and all the functions that relate them.
- b. Carefully name and describe all three variables.
- c. Write two functions. One relates the first variable to the second, and the other relates the second variable to the third.
- d. Put them together into a composite function that shows me how to get directly from the third variable to the first variable.
- e. Using a sample number, write a (word problem!) question and use your composite function to find the answer.
3. Here is the algorithm for converting the temperature from Celsius to Fahrenheit. First, multiply the Celsius temperature by $\frac{9}{5}$. Then, add 32.
- a. Write this algorithm as a mathematical function: Celsius temperature (C) goes *in*, Fahrenheit temperature (F) comes *out*. $F(C) = \underline{\hspace{2cm}}$
- b. Write the inverse
- c. Write a real-world word problem that you can solve by using that inverse function. (This does not have to be elaborate, but it has to show that you know what the inverse function does
- d. Use the inverse function that you found in part (b) to answer the question you asked in part (c).
4. $f(x) = \sqrt{x+1}$ $g(x) = \frac{1}{x}$. For a-e, I am *not* looking for answers like $[g(x)]^2$. Your answers should not have a g or an f in them, just a bunch of .
- a. $f(g(x)) =$
- b. $g(f(x)) =$
- c. $f(f(x)) =$
- d. $g(g(x)) =$
- e. $g(f(g(x))) =$
- f. What is the domain of $f(x)$?
- g. What is the domain of $g(x)$?
5. $f(x) = 20 - x$
- a. What is the domain?
- b. What is the inverse function?
- c. Test your inverse function. (No credit for just the words "it works"—I have to see your test
6. $f(x) = 3 + \frac{x}{7}$
- a. What is the domain?
- b. What is the inverse function?
- c. Test your inverse function. (Same note as above.)

7. $f(x) = \frac{2x}{3x-4}$

- What is the domain?
- What is the inverse function?
- Test your inverse function. (Same note.)

8. For each of the following diagrams, indicate roughly what the slope is.

-
-
-

9. $6x + 3y = 10$

$y = mx + b$ format: _____

Slope: _____

y-intercept: _____

Graph it!

Extra credit *add* them, and when you *multiply* them, you get the same answer.

- If one of the numbers is 5, what is the other number?
- If one of the numbers is x , what is the other number? (Your answer will be a function of x .)
- What number could x be that would not have *any possible* other number to go with it?

CHAPTER **3**

Inequalities

Chapter Outline

3.1 **INEQUALITIES**

3.1 Inequalities

Name: _____

- a. $4 < 6$ (I think we can all agree on that, yes?)
- Add 4 to both sides of the equation. _____ Is it still true?
 - Add -4 to both sides of the (original) equation. _____ Is it still true?
 - Subtract 10 from both sides of the (original) equation. _____ Is it still true?
 - Multiply both sides of the (original) equation by 4. _____ Is it still true?
 - Divide both sides of the (original) equation by 2. _____ Is it still true?
 - Multiply both sides of the (original) equation by -3 . _____ Is it still true?
 - Divide -2 . _____ Is it still true?
 - In general: what operations, when performed on an inequality, reverse
- b. $2x + 3 < 7$
- Solve for x .
 - Draw a number line below, and show where the solution set to this problem is.
 - Pick an x -value which, according to your drawing, is *inside* the solution set. Plug it into the original inequality $2x + 3 < 7$. Does the inequality hold true?
 - Pick an x -value which, according to your drawing, is *outside* the solution set. Plug it into the original inequality $2x + 3 < 7$. Does the inequality hold true?
- c. $10 - x \geq 4$
- Solve for x . Your first step should be adding x to both sides, so in your final equation, x is on the right side.
 - Solve for x again from the original equation. This time, leave x on the left side.
 - Did your two answers come out the same?
 - Draw a number line below, and show where the solution set to this problem is.
 - Pick an x -value which, according to your drawing, is *inside* the solution set. Plug it into the original inequality $10 - x \geq 4$. Does the inequality hold true?
 - Pick an x -value which, according to your drawing, is *outside* the solution set. Plug it into the original inequality $10 - x \geq 4$. Does the inequality hold true?
- d. $x = \pm 4$
- Rewrite this statement as two different statements, joined by “and” or “or.”
 - Draw a number line below, and show where the solution set to this problem is.
- e. $-3 < x \leq 6$
- Rewrite this statement as two different statements, joined by “and” or “or.”
 - Draw a number line below, and show where the solution set to this problem is.
- f. $x > 7$ or $x < -3$
- Draw a number line below, and show where the solution set to this problem is.
- g. $x > 7$ and $x < -3$
- Draw a number line below, and show where the solution set to this problem is.
- h. $x < 7$ or $x > -3$
- Draw a number line below, and show where the solution set to this problem is.

- i. $x > \pm 4$
- Rewrite this statement as two different statements, joined by “and” or “or.”
 - Draw a number line below, and show where the solution set to this problem is.

Name: _____

Homework: Inequalities

1. $2x + 7 \leq 4x + 4$

- Solve for x .
- Draw a number line below, and show where the solution set to this problem is.
- Pick an x -value which, according to your drawing, is *inside* the solution set. Plug it into the original inequality $2x + 7 \leq 4x + 4$. Does the inequality hold true?
- Pick an x -value which, according to your drawing, is *outside* the solution set. Plug it into the original inequality $2x + 7 \leq 4x + 4$. Does the inequality hold true?

2. $14 - 2x < 20$

- Solve for x .
- Draw a number line below, and show where the solution set to this problem is.
- Pick an x -value which, according to your drawing, is *inside* the solution set. Plug it into the original inequality $14 - 2x < 20$. Does the inequality hold true?
- Pick an x -value which, according to your drawing, is *outside* the solution set. Plug it into the original inequality $14 - 2x < 20$. Does the inequality hold true?

3. $-10 < 3x + 2 \leq 5$

- Solve for x .
- Draw a number line below, and show where the solution set to this problem is.
- Pick an x -value which, according to your drawing, is *inside* the solution set. Plug it into the original inequality $-10 < 3x + 2 \leq 5$. Does the inequality hold true?
- Pick an x -value which, according to your drawing, is *outside* the solution set. Plug it into the original inequality $-10 < 3x + 2 \leq 5$. Does the inequality hold true?

4. $x < 3$ and $x < 7$. Draw a number line below, and show where the solution set to this problem is.

5. $x < 3$ or $x < 7$. Draw a number line below, and show where the solution set to this problem is.

6. $x - 2y \geq 4$

- Solve for y .
- Now—for the moment—let’s pretend that your equation said equals—intercept of that line, and graph it.

Slope: _____

y -Intercept: _____

- Now, pick any point (x, y) that is *above* that line. Plug the x and y coordinates into your inequality from part (a). Does this point fit the inequality? (Show your work...)
 - Now, pick any point (x, y) that is *below* that line. Plug the x and y coordinates into your inequality from part (a). Does this point fit the inequality? (Show your work...)
 - So, is the solution to the inequality the points below/above the line? Shade the appropriate region on your graph.
7. Using a similar technique, draw the graph of $y \geq x^2$. (If you don’t remember what the graph of $y = x^2$ looks like,

try plotting a few points!)

Name: _____

Inequality Word Problems

- Jacob is giving a party. 20 people showed up, but he only ordered 4 pizzas! Fortunately, Jacob hasn't *cut* the pizzas yet. He is going to cut each pizza into n slices, and he needs to make sure there are enough slices for everyone at the party to get at least one. Write an inequality or set that describes what n has to be.
- Whitney wants to drive to Seattle. She needs 100 gallons of gas to make the trip, but she has only \$80 allocated for gas. Her strategy is to wait until the price of gas is low enough that she can make the trip. Write an inequality or set that describes what the price of gas *has to be* for Whitney to be able to reach Seattle. Be sure to clearly define your variable(s)!
- Your evil math teacher, who shall go nameless, is only giving two tests for the whole grading period. They count equally—your grade will be the average of the two. Your first test was a 90. Write an inequality or set that describes what your *second* test grade has to be, in order for you to bring home an A on your report card. ("A" means 93 or above.) Be sure to clearly define your variable(s)!
- Laura L is going to build a movie theater with n screens. At each screen, there will be 200 seats for the audience to watch that movie. (So maximum capacity is 200 audience members per screen.) In addition to audience members, there are 20 employees on the premises at any given time (selling tickets and popcorn and so on). According to code (which I am making up), she must have at least one bathroom for each 100 people in the building. (Of course, it's fine to build more bathrooms than that, if she wants!)
 - Write a function (this will be an equation) relating the number of screens (n) to the total number of people who can possibly be in the building (p). Which one is dependent? Which one is independent?
 - Write an inequality(p) to the number of bathrooms (b).
 - Now write a composite inequality
- Make up your own word problem for which the solution is an inequality, and solve it. The topic should be breakfast

Name: _____

Absolute Value Equations

- $|4| =$
- $|-5| =$
- $|0| =$
- OK, now, I'm thinking of a number. All I will tell you is that the absolute value of my number is 7.
 - Rewrite my question as a math equation instead of a word problem.
 - What can my number be?
- I'm thinking of a different number. This time, the absolute value of my number is 0.
 - Rewrite my question as a math equation instead of a word problem.
 - What can my number be?
- I'm thinking of a different number. This time, the absolute value of my number is -4 .
 - Rewrite my question as a math equation instead of a word problem.
 - What can my number be?

7. I'm thinking of a different number. This time, the absolute value of my number is less than 7.

- Rewrite my question as a math inequality instead of a word problem.
- Does 8 work?
- Does 6 work?
- Does -8 work?
- Does -6 work?
- Write an inequality that describes all possible values for my number.

8. I'm thinking of a different number. This time, the absolute value of my number is greater than 4.

- Rewrite my question as a math inequality instead of a word problem.
- Write an inequality that describes all possible values for my number. (Try a few numbers, as we did in 7.)

9. I'm thinking of a different number. This time, the absolute value of my number is greater than -4 .

- Rewrite my question as a math inequality instead of a word problem.
- Write an inequality that describes all possible values for my number.

Stop at this point and check your answers with me before going on to the next side

10. $|x + 3| = 7$

- First, forget that it says, $|x + 3| = 7$ and just think of it as "a number." The absolute value of *this number* is 7. So what can *this number* be?
- Now, remember that "this number" is $x + 3$. So write an equation that says that $x + 3$ can be .
- Solve the equation(s) to find what x can be.
- Plug your answer(s) back into the original equation $|x + 3| = 7$ and see if they work.

11. $4|3x - 2| + 5 = 17$

- This time, because the absolute value is not alone, we're going to start with some algebra. Leave $|3x - 2|$ alone, but get rid of everything *around* it, so you end up with $|3x - 2|$ alone on the left side, and some other number on the right.
- Now, remember that "some number" is $3x - 2$. So write an equation that says that $3x - 2$ can be .
- Solve the equation(s) to find what x can be.
- Plug your answer(s) back into the original equation $4|3x - 2| + 5 = 17$ and see if they work.

12. $|x - 3| + 5 = 4$

- Solve, by analogy to the way you solved the last two problems.
- Plug your answer(s) back into the original equation $|x - 3| + 5 = 4$ and see if they work.

13. $|x - 2| = 2x - 10$.

- Solve, by analogy to the way you solved the last two problems.
- Plug your answer(s) back into the original equation $|x - 2| = 2x - 10$ and see if they work.

Name: _____

Homework: Absolute Value Equations

- $|x| = 5$
 - Solve for x
 - Check your answer(s) in the original equation.

- b. $|x|=0$
- Solve for x
 - Check your answer(s) in the original equation.
- c. $|x|=-2$
- Solve for x
 - Check your answer(s) in the original equation.
- d. $10|x|=5$
- Solve for x
 - Check your answer(s) in the original equation.
- e. $|x+3|=1$
- Solve for x
 - Check your answer(s) in the original equation.
- f. $\frac{4|x-2|}{3}=2$
- Solve for x
 - Check your answer(s) in the original equation.
- g. $7|2x+3|-4=4$
- Solve for x
 - Check your answer(s) in the original equation.
- h. $|2x-3|=x$
- Solve for x
 - Check your answer(s) in the original equation.
- i. $|2x+2|=x$
- Solve for x
 - Check your answer(s) in the original equation.
- j. $|x-5|=2x-7$
- Solve for x
 - Check your answer(s) in the original equation.

Check-yourself hint: for numbers 8, 9, and 10, one of them has no valid solutions, one has one valid solution, and one has two valid solutions.

Name: _____

Absolute Value Inequalities

- a. $|x|\leq 7$
- Solve.
 - Graph your solution on a number line
 - Choose a point that is
 - Choose a point that is not
- b. $|2x+3|\leq 7$
- Write down the solution for what $2x+3$ has to be. This should look exactly like your answer to number 1, except with a $(2x+3)$ instead of an (x) .

- b. Now, solve that inequality for x .
 - c. Graph your solution on a number line
 - d. Choose a point that is
 - e. Choose a point that is not
- c. $4|3x - 6| + 7 > 19$
- a. Solve for $|3x - 6|$. (That is, leave the $|3x - 6|$ part alone, but get rid of all the stuff around it.)
 - b. Write down the inequality for what $(3x - 6)$ has to be.
 - c. Now, solve that inequality for x .
 - d. Graph your solution on a number line
 - e. Choose a point that is
 - f. Choose a point that is not
- d. $\frac{|3x-4|}{2} + 6 < 3$
- a. Solve for x . (You know the drill by now!)
 - b. Graph your solution on a number line
 - c. Choose a point that is
 - d. Choose a point that is not
- e. $6|2x^2 - 17x - 85| + 5 \geq 3$
- a. Solve for x .
 - b. Graph your solution on a number line
 - c. Choose a point that is
 - d. Choose a point that is not

Name: _____

Homework—Absolute Value Inequalities

- a. $|4 + 3x| = 2 + 5x$ (... OK, this isn't an *inequality*, but I figured you could use a bit more practice at these)
- b. $|x| = x - 1$
- c. $4|2x - 3| - 5 \geq 3$
 - a. Solve for x .
 - b. Graph your solution on a number line
 - c. Choose a point that is
 - d. Choose a point that is not
- d. $3|x - 5| + 2 < 17$
 - a. Solve for x .
 - b. Graph your solution on a number line
 - c. Choose a point that is
 - d. Choose a point that is not
- e. $-3|x - 5| + 2 < 17$
 - a. Solve for x .
 - b. Graph your solution on a number line
 - c. Choose a point that is
 - d. Choose a point that is not
- f. $2|x + 2| + 6 < 6$
 - a. Solve for x .
 - b. Graph your solution on a number line
 - c. Choose a point that is

- d. Choose a point that is not

Name: _____

Graphing Inequalities and Absolute Values

a. $9x + 3y \leq 6$

- Put into a sort of $y = mx + b$ format, except that it will be an *inequality*.
- Now, ignore the fact that it is an inequality—pretend it is a line, and graph that line.
- Now, to graph the inequality, shade in the area either *above*/*below* the line, as appropriate. (Hint: does y have to be *less than* the values on the line, or *greater than* them?)
- Test your answer. Choose a point (any point) in the region you shaded, and test it in the inequality. Does the inequality work? (Show your work.)
- Choose a point (any point) in the region you did not

b. $4x \leq 2y + 5$

- Graph the inequality, using the same steps as above.
- Test your answer by choosing one point *in* the shaded region. Do they give you the answers they should? (Show your work.)

c. $y = |x|$

- Create a table of points. Your table should include at least two positive x -values, two negative x -values, and $x = 0$.
- Graph those points, and then draw the function.

d. $y = |x| + 3$. Graph this *without* a table of points, by remembering what “adding 3” does to any graph. (In other words, what will these y -values be like compared to your y -values in 3?)

e. $y = -|x|$. Graph this *without* a table of points, by remembering what “multiplying by -1 ” does to any graph. (In other words, what will these y -values be like compared to your y -values in 3?)

f. Now, let’s put it all together!!!

- Graph it. $y = -|x| + 2$
- Graph $y \geq -|x| + 2$. Your answer will *either* be a shaded region on a 2-dimensional graph, *or* on a number line.
- Test your answer by choosing one point *in* the shaded region. Do they give you the answers they should? (Show your work.)
- Graph $-|x| + 2 < 0$. Your answer will *either* be a shaded region on a 2-dimensional graph, *or* on a number line.
- Test your answer by choosing one point *in* the shaded region. Do they give you the answers they should? (Show your work.)

g. Extra for experts: $y \geq 3|x + 4|$

- Graph it. Think hard about what that $+4$ and that 3 will do. Generate a few points if it will help you!
- Test your answer by choosing one point *in* the shaded region. Do they give you the answers they should? (Show your work.)

Name: _____

Homework: Graphing Inequalities and Absolute Values

1. The famous detectives Guy Noir and Nick Danger are having a contest to see who is better at catching bad guys. At 8:00 in the evening, they start prowling the streets of the city. They have twelve hours. Each of them gets 10

points for every purse-snatcher he catches, and 15 points for every cat-burglar. At 8:00 the next morning, they meet in one of their dingy offices to compare notes. “I got 100 points,” brags Nick. If Guy gets enough snatchers and burglars, he will win the contest.

- Label and clearly describe the relevant variables.
- Write an inequality relating the variables you listed in part (a). I should be able to read it as, “If this inequality is true, then Guy wins the contest.”
- Graph the inequality from part (b).

2. The graph below shows the function $y = f(x)$.

- Graph $y \leq f(x)$. Your answer will *either* be a shaded region on a 2–dimensional graph, *or* on a number line.
- Graph $f(x) < 0$. Your answer will *either* be a shaded region on a 2–dimensional graph, *or* on a number line.

3. $x - 2y > 4$

- Graph.
- Pick a point in your shaded region, and plug it back into our original equation $x - 2y > 4$. Does the inequality work? (Show your work!)
- Pick a point which is not $x - 2y > 4$. Does the inequality work? (Show your work!)

4. $|x| - y \geq 2$

- Graph.
- Pick a point in your shaded region, and plug it back into our original equation $|x| - y \geq 2$. Does the inequality work? (Show your work!)
- Pick a point which is not $|x| - y \geq 2$. Does the inequality work? (Show your work!)

5. $y > x^3$

- Graph. (Plot points to get the shape.)
- Pick a point in your shaded region, and plug it back into our original equation $y > x^3$. Does the inequality work? (Show your work!)
- Pick a point which is not $y > x^3$. Does the inequality work? (Show your work!)

6. Graph: $y + |x| < -|x|$. Think hard—you can do it!

Name: _____

Sample Test: Inequalities and Absolute Values

- $1 < 4 - 3x \leq 10$
 - Solve for x .
 - Draw a number line below, and show where the solution set to this problem is.
 - Pick an x -value which, according to your drawing, is *inside* the solution set. Plug it into the original inequality $1 < 4 - 3x \leq 10$. Does the inequality hold true? (Show your work!)
 - Pick an x -value which, according to your drawing, is *outside* the solution set. Plug it into the original inequality $1 < 4 - 3x \leq 10$. Does the inequality hold true? (Show your work!)
- Find the x value(s) that make this equation true: $4|2x + 5| - 3 = 17$
- Find the x value(s) that make this equation true: $|5x - 23| = 21 - 6x$
- $\frac{|2x-3|}{3} + 7 > 9$
 - Solve for x .
 - Show graphically where the solution set to this problem is.
- $-3|x + 4| + 7 \geq 7$

- a. Solve for x .
 - b. Show graphically where the solution set to this problem is.
- f. Make up and solve an inequality word problem having to do with hair
- a. Describe the scenario in words.
 - b. Label and clearly describe the variable or variables.
 - c. Write the inequality. (Your answer here should be completely determined)
- g. Graph $y \geq -|x|+2$
- h. $|2y|-|x|> 6$
- a. Rewrite this as an inequality with no absolute values, *fourth quadrant* (lower-right-hand corner of the graph).
 - b. Graph what this looks like, in the fourth quadrant only.
- i. Graph $y = x - |x|$

CHAPTER

4

Simultaneous Equations

Chapter Outline

4.1 SIMULTANEOUS EQUATIONS

4.1 Simultaneous Equations

Name: _____

Distance, Rate, and Time

1. You set off walking from your house at 2 miles per hour.

a. Fill in the following table.

| | | | | | |
|--------------------------------|--------------------|--------|---------|---------|----------|
| After this much time (t) | $\frac{1}{2}$ hour | 1 hour | 2 hours | 3 hours | 10 hours |
| You have gone this far (d) | | | | | |

b. Write the function $d(t)$.

2. You set off driving 60 miles per hour.

a. Fill in the following table.

| | | | | | |
|--------------------------------|--------------------|--------|---------|---------|----------|
| After this much time (t) | $\frac{1}{2}$ hour | 1 hour | 2 hours | 3 hours | 10 hours |
| You have gone this far (d) | | | | | |

b. Write the function $d(t)$.

3. You set off in a rocket, flying upward at 200 miles per hour.

a. Fill in the following table.

| | | | | | |
|--------------------------------|--------------------|--------|---------|---------|----------|
| After this much time (t) | $\frac{1}{2}$ hour | 1 hour | 2 hours | 3 hours | 10 hours |
| You have gone this far (d) | | | | | |

b. Write the function $d(t)$.

4. Write the general relationship between distance traveled (d), rate (r), and time (t).

5. You start off for school at 55 mph. $\frac{1}{5}$ of an hour later, your mother realizes you forgot your lunch. She dashes off after you, at 70 mph. Somewhere on the road, she catches up with you, throws your lunch from her car into yours, and vanishes out of sight.

Let d equal the distance from your home where your mother catches up with you. Let t equal the time that you took to reach that distance. (Note that you and your mother traveled the *same distance*, but in *different time*.) d should be measured in miles, and t in hours (not minutes).

- a. Write the distance-rate-time relationship for you, from the time you leave the house until your mother catches up with you.
- b. Write the distance-rate-time relationship for your mother, from the time she
- c. Based on those two equations, can you figure out how far you were from the house when your mother caught you?

Name: _____

Homework: Simultaneous Equations by Graphing

In each problem, find all

1. $2y = 6x + 10$ and $3y = 12x + 9$

- a. Put both equations into $y = mx + b$ format. Then graph them.
- b. List all points of intersection.
- c. Check these points to make sure they satisfy both equations.

2. $y = 2x - 3$ and $3y = 6x + 3$

- a. Put both equations into $y = mx + b$ format. Then graph them.
- b. List all points of intersection.
- c. Check these points to make sure they satisfy both equations.

3. $y = x - 3$ and $2y = 2x - 6$

- a. Put both equations into $y = mx + b$ format. Then graph them on the back.
- b. List all points of intersection.
- c. Check these points to make sure they satisfy both equations.

4. $y = x$ and $y = x^2 - 1$

- a. Graph them both on the back.
- b. List all points of intersection.
- c. Check these points to make sure they satisfy both equations.

5. $y = x^2 + 2$ and $y = x$

- a. Graph them both on the back.
- b. List all points of intersection.
- c. Check these points to make sure they satisfy both equations.

6. $y = x^2 + 4$ and $y = 2x + 3$

- a. Put the second equation into $y = mx + b$ format. Then graph them both on the back.
- b. List all points of intersection.
- c. Check these points to make sure they satisfy both equations.

7. Time for some generalizations. . .

- a. When graphing two lines,
- b. When graphing a line and a parabola,

This last problem does not involve two lines, or a line and a parabola: it's a bit weirder than that. It is the only problem on this sheet that should require a calculator

8. $y = \frac{6x}{x^2+1}$ and $y = 4\sqrt{x} - 5$

- Graph them both on your calculator and find the point of intersection as accurately as you can.
- Check this point to make sure it satisfies both equations.

Name: _____

Simultaneous Equations

- Emily is hosting a major after-school party. The principal has imposed two restrictions. First (because of the fire codes) the total number of people attending (teachers and students combined) must be 56. Second (for obvious reasons) there must be one teacher for every seven students. How many students and how many teachers are invited to the party?
 - Name and clearly identify the variables.
 - Write the equations that relate these variables.
 - Solve. Your final answers should be complete English sentences (not “the answer is 2,” but “there were 2 students there.” Except it won’t be 2. You get the idea, right?)
- A group of 75 civic-minded students and teachers are out in the field, picking sweet potatoes for the needy. Working in the field, Kasey picks three times as many sweet potatoes as Davis—and then, on the way back to the car, she picks up five *more* sweet potatoes than that! Looking at her newly increased pile, Davis remarks “Wow, you’ve got 29 more potatoes than I do!” How many sweet potatoes did Kasey and Davis each pick?
 - Name and clearly identify the variables.
 - Write the equations that relate these variables.
 - Solve. Your final answers should be complete English sentences.
- A hundred ants are marching into an anthill at a slow, even pace of 2 miles per hour. Every ant is carrying *either* one bread crumb, *or* two pieces of grass. There are 28 more bread crumbs than there are pieces of grass. How many of each are there?
 - Name and clearly identify the variables.
 - Write the equations that relate these variables.
 - Solve. Your final answer should be a complete English sentence.
- Donald is 14 years older than Alice. 22 years ago, she was only half as old as he was. How old are they today?
 - Name and clearly identify the variables.
 - Write the equations that relate these variables.
 - Solve. Your final answers should be complete English sentences.
- Make up your own word problem like the ones above, and solve it.
- $3x - 2y = 16$
 $7x - y = 30$
 - Solve by substitution
 - Solve by elimination
 - Check your answer
- $3x + 2y = 26$
 $2x + 4y = 32$
 - Solve by substitution
 - Solve by elimination
 - Check your answer
- Under what circumstances is substitution easiest?
- Under what circumstances is elimination easiest?

Name: _____

Homework: Simultaneous Equations

a. $6x + 2y = 6$
 $x - y = 5$

- Solve by substitution.
- Check your answers.

b. $3x + 2y = 26$
 $2x - 4y = 4$

- Solve by elimination.
- Check your answers.

c. $2y - x = 4$
 $2x + 20y = 4$

- Solve any way you like.
- Check your answers.

d. $3x + 4y = 12$
 $5x - 3y = 20$

- Solve any way you like.
- Check your answers.

e. $ax + y = 6$
 $2x + y = 4$

- Solve any way you like. You are solving for x and y ; a is just a constant. (So your final answer will say $x = \text{blah} - \text{blah}$, $y = \text{blah} - \text{blah}$." The *blah-blah* will both have a in them.)
- Check your answers.

The “Generic” Simultaneous Equations

Here is the generic simultaneous equations

$$\begin{aligned} ax + by &= e \\ cx + dy &= f \end{aligned}$$

I call them “generic” because every possible pair of simultaneous equations looks exactly like that, except with numbers instead of a, b, c, d, e , and f . We are going to solve these equations.

Very important!!! $x = \text{blah} - \text{blah}$ where the *blah-blah* has only a, b, c, d, e , and f : no x or y . And, of course, $y =$ some *different* formula with only a, b, c, d, e , and f . If we can do that, we will be able to use these formulas to immediately solve *any* pair of simultaneous equations, just by plugging in the numbers.

We can solve this by elimination or by substitution. I am going to solve for y by elimination. I will use all the exact same steps we have always used in class.

Step 1: Make the coefficients of x line up

To do this, I will multiply the top equation by c and the bottom equation by a .

$$\begin{aligned} acx + bcy &= ec \\ acx + ady &= af \end{aligned}$$

Step 2: Subtract the second equation from the first

This will make the x terms go away.

$$\begin{aligned} acx + bcy &= ec \\ -(acx + ady) &= af \\ \hline bcy - ady &= ec - af \end{aligned}$$

Step 3: Solve for y

This is something we've done many times in class, right? First, pull out a y ; then divide by what is in parentheses.

$$\begin{aligned} bcy - ady &= ec - af \\ y(bc - ad) &= ec - af \\ y &= \frac{ec - af}{bc - ad} \end{aligned}$$

So what did we do?

We have come up with a totally generic formula for finding y in any simultaneous equations. For instance, suppose we have...

$$\begin{aligned} 3x + 4y &= 18 \\ 5x + 2y &= 16 \end{aligned}$$

We now have a new way of solving this equation: just plug into $y = \frac{ec-af}{bc-ad}$. That will tell us that $y = \frac{(18)(5)-(3)(16)}{(4)(5)-(3)(2)} = \frac{90-48}{20-6} = \frac{42}{14} = 3$.

Didja get it?

Here's how to find out.

- Do the whole thing again, starting with the generic simultaneous equations, except solve for x instead of y .
- Use your formula to find x in the two equations I did at the bottom (under "So what did we do?")
- Test your answer by plugging your x and my $y = 3$ into those equations to see if they work!

Name: _____

Sample Test: 2 Equations and 2 Unknowns

- Evan digs into his pocket to see how much pizza he can afford. He has \$3.00, exactly enough for two slices. But it is *all* in dimes and nickels! Counting carefully, Evan discovers that he has twice as many dimes as nickels.
 - Identify and clearly label the variables.
 - Write two equations that represent the two statements in the question.
 - Solve these equations to find how many nickels and dimes Jeremy has.
- Black Bart and the Sheriff are having a gunfight at high noon. They stand back to back, and start walking away from each other: Bart at 4 feet per second, the Sheriff at 6 feet per second. When they turn around to shoot, they find that they are 55 feet away from each other.
 - Write the equation $d = rt$ for Bart.

- b. Write the equation $d = rt$ for the Sheriff.
- c. Solve, to answer the question: for how long did they walk away from each other?
- d. How far did Bart walk?

3. Mrs. Verbatim the English teacher always assigns 5 short stories (2,000 words each) for every novel (60,000 words each) that she assigns. This year she has decided to assign a total of 350,000 words of reading to her students. How many books and how many short stories should she select?

- a. Identify and clearly label the variables.
- b. Write two equations that represent the two conditions that Mrs. V imposed.
- c. Solve these equations to find the number of works she will be assigning.
4. Solve by graphing

$$y = x^2 - 3$$

$$y = -|x| + 2$$

5. Solve, using substitution

$$3x + y = -2$$

$$6x - 2y = 12$$

6. Solve, using elimination

$$2x + 3y = -11$$

$$3x - 6y = 4\frac{1}{2}$$

7. Solve any way you like.

$$2x = 6y + 12$$

$$x - 9 = 3y$$

8. Solve any way you like.

$$2y + 3x = 20$$

$$\frac{1}{2}y + x = 6$$

9. a. Solve for x . (*No credit without showing your work!)

$$ax + by = e$$

$$cx + dy = f$$

b. Use the formula you just derived to find x in these equations.

$$3x + 4y = 7$$

$$2x + 3y = 11$$

Redo number 9. If you used elimination before, use substitution. If you used substitution, use elimination.

CHAPTER **5**

Quadratic Functions

Chapter Outline

5.1 QUADRATIC FUNCTIONS

5.1 Quadratic Functions

Name: _____

Multiplying Binomials

or, “These are a few of my favorite formulae”

1. Multiply: $(x + 2)(x + 2)$

Test your result by plugging $x = 3$ into both my original function, and your resultant function. Do they come out the same?

2. Multiply: $(x + 3)(x + 3)$

Test your result by plugging $x = -1$ into both my original function, and your resultant function. Do they come out the same?

3. Multiply: $(x + 5)(x + 5)$

Test your result by plugging $x = \frac{1}{2}$ into both my original function, and your resultant function. Do they come out the same?

4. Multiply: $(x + a)(x + a)$

Now, leave x as it is, but plug $a = 3$ into both my original function, and your resultant function. Do you get two functions that are *equal*? Do they look familiar?

5. Do not $(x + a)^2$ that you found in number 4.

a. $(x + 4)(x + 4)$

b. $(y + 7)^2$

c. $(z + \frac{1}{2})^2$

d. $(m + \sqrt{2})^2$

e. $(x - 3)^2$ (*so in this case, a is -3 .)

f. $(x - 1)^2$

g. $(x - a)^2$

6. Earlier in class, we found the following generalization: $(x + a)(x - a) = x^2 - a^2$. Just to refresh your memory on how we found that, test this generalization for the following cases.

a. $x = 10, a = 0$

b. $x = 10, a = 1$

c. $x = 10, a = 2$

d. $x = 10, a = 3$

7. Test the same generalization by multiplying out $(x + a)(x - a)$ explicitly.

8. Now, use that “difference between two squares” generalization. As in number 5, do not solve these by multiplying them out, but by plugging appropriate values into the generalization in number 6.

- a. $(20 + 1)(20 - 1) =$
 b. $(x + 3)(x - 3) =$
 c. $(x + \sqrt{2})(x - \sqrt{2})$
 d. $(x + \frac{2}{3})(x - \frac{2}{3})$

Name: _____

Homework: Multiplying Binomials

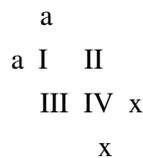
Memorize these:

$$(x + a)^2 = x^2 + 2ax + a^2$$

$$(x - a)^2 = x^2 - 2ax + a^2$$

$$x^2 - a^2 = (x + a)(x - a)$$

1. In the following drawing, one large square is divided into four regions. The four small regions are labeled with Roman numerals because I like to show off.



- a. How long is the left side of the entire figure? _____
 b. How long is the bottom of the entire figure? _____
 c. One way to compute the area of the entire figure is to multiply these two numbers (total height times total width). Write this product down here: Area = _____
 d. Now: what is the area of the small region labeled I? _____
 e. What is the area of the small region labeled II? _____
 f. What is the area of the small region labeled III? _____
 g. What is the area of the small region labeled IV? _____
 h. The other way to compute the area of the entire figure is to add up these small regions. Write this sum down here: Area = _____
 i. Obviously, the answer to (c) and the answer to (h) have to be the same, since they are both the area of the entire figure. So write down the equation setting these two equal to each other here: _____
 j. Does that look like one of our three formulae?

2. Multiply these out “manually.”

- a. $(x + 3)(x + 4)$
 b. $(x + 3)(x - 4)$
 c. $(x - 3)(x - 4)$
 d. $(2x + 3)(3x + 2)$
 e. $(x - 2)(x^2 + 2x + 4)$

f. Check your answer to part (e) by substituting the number 3 into both my original function, and your answer. Do they come out the same?

3. Multiply these out using the formulae above.

a. $(x + \frac{3}{2})^2$

b. $(x - \frac{3}{2})^2$

c. $(x + 3)^2$

d. $(3 + x)^2$

e. $(x - 3)^2$

f. $(3 - x)^2$

g. Hey, why did (e) and (f) come out the same? ($x - 3$ isn't the same as $3 - x$, is it?)

h. $(x + \frac{1}{2})(x + \frac{1}{2})$

i. $(x + \frac{1}{2})(x - \frac{1}{2})$

j. $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$

k. Check your answer to part (j) by running through the whole calculation on your calculator: $\sqrt{5} - \sqrt{3} = \underline{\hspace{2cm}}$, $\sqrt{5} + \sqrt{3} = \underline{\hspace{2cm}}$, multiply them and you get $\underline{\hspace{2cm}}$.

4. Now, let's try going backward. $(x + \text{something})^2$, or as $(x - \text{something})^2$, or as $(x + \text{something})(x - \text{something})$. In each case, check your answer by multiplying back to see if you get the original expression.

a. $x^2 - 8x + 16 = \underline{\hspace{2cm}}$

Check by multiplying back: $\underline{\hspace{2cm}}$

b. $x^2 - 25 = \underline{\hspace{2cm}}$

Check by multiplying back: $\underline{\hspace{2cm}}$

c. $x^2 + 2x + 1 = \underline{\hspace{2cm}}$

Check by multiplying back: $\underline{\hspace{2cm}}$

d. $x^2 - 20x + 100 = \underline{\hspace{2cm}}$

Check by multiplying back: $\underline{\hspace{2cm}}$

e. $4x^2 - 9 = \underline{\hspace{2cm}}$

Check by multiplying back: $\underline{\hspace{2cm}}$

5. Enough squaring: let's go one higher, and see what $(x + a)^3$ is!

a. $(x + a)^3$ means $(x + a)(x + a)(x + a)$. You already know what $(x + a)(x + a)$ is. So multiply *that* by $(x + a)$ to find the cubed formula. (Multiply term-by-term, then collect like terms.)

b. Use the formula you just found $(y + 3)^3$

c. Use the same formula to find $(y - 3)^3$.

Name: $\underline{\hspace{2cm}}$

Factoring

The first step is always to “pull out” as much as you can

1. Multiply the following, using the distributive property:

$$3x(4x^2 + 5x + 2) = \underline{\hspace{4cm}}$$

2. Now, you're going to do the same thing backward.

a. "Pull out" the common term of $4y$ from the following expression.

$$16y^3 + 4y^2 + 8y = 4y (\underline{\hspace{4cm}})$$

b. Check yourself, by multiplying $4y$ by the term you put in parentheses.

c. Did it work?

For each of the following expressions, pull out the highest common factor you can find.

3. $9xy + 12x = \underline{\hspace{4cm}}$

4. $10x^2 + 9y^2 = \underline{\hspace{4cm}}$

5. $100x^3 + 25x^2 = \underline{\hspace{4cm}}$

6. $4x^2y + 3y^2x = \underline{\hspace{4cm}}$

Next, look to apply our three formulae

Factor the following by using our three formulae for $(x + y)^2$, $(x - y)^2$, and $x^2 - y^2$.

7. $x^2 - 9 = \underline{\hspace{4cm}}$

8. $x^2 - 10x + 25 = \underline{\hspace{4cm}}$

9. $x^2 + 8x + 16 = \underline{\hspace{4cm}}$

10. $x^2 + 9 = \underline{\hspace{4cm}}$

11. $3x^2 - 27 = \underline{\hspace{4cm}}$ (*hint: start by pulling out the common factor!*)

If all else fails, factor the "old-fashioned way"

12. $x^2 + 7x + 10 = \underline{\hspace{4cm}}$

Check your answer by multiplying back:

13. $x^2 - 5x + 6 = \underline{\hspace{4cm}}$

Check your answer by plugging a number into the original expression, and into your modified expression:

14. $x^2 - 6x + 5 = \underline{\hspace{4cm}}$

15. $x^2 + 8x + 6 = \underline{\hspace{4cm}}$

16. $x^2 - x - 12 = \underline{\hspace{4cm}}$

17. $x^2 + x - 12 = \underline{\hspace{4cm}}$

18. $x^2 + 4x - 12 = \underline{\hspace{4cm}}$

19. $2x^2 + 7x + 12 = \underline{\hspace{4cm}}$

Name:

Homework: Factah Alla Dese Heah Sprexions

1. $3x^3 + 15x^2 + 18x$

Check your answer by multiplying back:

2. $x^2 - 12x + 32$

Check your answer by plugging the number 4 into both my original expression, and your factored expression. Did they come out the same?

Is there any number x for which they would *not* come out the same?

3. $x^2 - 4x - 32$

4. $8x^2 - 18y^4$

5. $x^2 + 18x + 32$

6. $x^2 - 18x + 32$

7. $2x^2 + 12x + 10$

8. $100x^2 + 800x + 1000$

9. $x^2 + 4x - 32$

10. $x^4 - y^4$

11. $x^2 + 13x + 42$

12. $x^2 + 16$

13. $4x^2 - 9$

14. $3x^2 - 48$

15. $2x^2 + 10x + 12$

16. $x^2 - 9$

17. $2x^2 + 11x + 12$

Name: _____

Introduction to Quadratic Equations

For problems

1. $x + y = 0$

2. $xy = 0$

3. $xy = 1$

4. $xy > 0$

5. $xy < 0$

6. OK, here's a different sort of problem. A swimming pool is going to be built, 3 yards long by 5 yards wide. Right outside the swimming pool will be a tiled area, which will be the same width all around. The total area of the swimming pool plus tiled area must be 35 yards.

a. Draw the situation. This doesn't have to be a fancy drawing, just a little sketch that shows the 3, the 5, and the unknown width of the tiled area.

b. Write an algebraic equation that gives the unknown width of the tiled area.

c. Solve that equation to find the width.

d. Check your answer—does the whole area come to 35 yards?

Solve for

7. $x^2 + 5x + 6 = 0$

Check your answers by plugging them into the original equation. Do they both work?

8. $2x^2 - 16x + 15 = 0$

9. $x^3 + 4x^2 - 21x = 0$

10. $3x^2 - 27 = 0$

Solve for

11. $x^2 = 9$

12. $(x - 4)^2 = 9$

13. $x^2 - 8x + 16 = 9$

14. $x^2 - 8x = -7$

Name: _____

Homework: Introduction to Quadratic Equations

If a ball is thrown up into the air, the equation for its position is:

$$h(t) = h_o + v_o t - 16t^2$$

where...

- h is the height—given as a function of time, of course—measured in feet.
- t is the time, measured in seconds.
- h_o is the initial height that it had when it was thrown—or, to put it another way, h_o is height when $t = 0$.
- v_o is the initial velocity that it had when it was thrown, measured in feet per second—or, to again put it another way, v_o is the velocity when $t = 0$.

This is sometimes called the equation of motion

Use that equation to solve the questions below.

- a. I throw a ball up from my hand. It leaves my hand 3 feet above the ground, with a velocity of 35 feet per second. (So these are the *initial* height and velocity, h_o and v_o .)
 - a. Write the equation of motion for this ball. You get this by taking the general equation I gave you above, and plugging in the specific h_o and v_o for this particular ball.
 - b. How high is the ball after two seconds? (In other words, what h value do you get when you plug in $t = 2$?)
 - c. What h value do you get when you plug $t = 0$ into the equation? Explain in words what this result means.
- b. I throw a different ball, much more gently. This one also leaves my hand 3 feet above the ground, but with a velocity of only 2 feet per second.
 - a. Write the equation of motion for this ball.
 - b. How long does it take the ball to reach the ground? (In other words, what t value will make h come out zero?)
 - c. How long does it take the ball to come back to my hand (which is still 3 feet above the ground)?
- c. A spring leaps up from the ground, and hits the ground again after 3 seconds. What was the velocity of the spring as it left the ground?
- d. I drop a ball from a 100 ft building. How long does it take to reach the ground?
- e. Finally, one straight equation to solve for x :

$$(x - 2)(x - 1) = 12$$

Name: _____

Completing the Square

Solve for

1. $x^2 = 18$

2. $x^2 = 0$

3. $x^2 = -60$

4. $x^2 + 8x + 12 = 0$

a. Solve by factoring.

b. Now, we're going to solve it a different

c. Now, the left side can be written as $(x + \text{something})^2$. Rewrite it that way, and then solve from there.

d. Did you get the same answers this way that you got by factoring?

Fill in the blanks

5. $(x - 3)^2 = x^2 - 6x + \underline{\quad}$

6. $(x - \frac{3}{2})^2 = x^2 - \underline{\quad}x + \underline{\quad}$

7. $(x + \underline{\quad})^2 = x^2 + 10x + \underline{\quad}$

8. $(x - \underline{\quad})^2 = x^2 - 18x + \underline{\quad}$

Solve for

9. $x^2 - 20x + 90 = 26$

10. $3x^2 + 2x - 4 = 0$ (*Hint:* start by dividing by 3. The x^2 term should never have a coefficient when you are completing the square.)

Name: _____

Homework: Completing the Square

1. A pizza (a perfect circle) has a 3 radius for the real pizza part (the part with cheese). But they advertise it as having an area of 25π square inches, because they include the crust. How wide is the crust?
2. According to NBA rules, a basketball court must be precisely 94 feet long and 50 feet wide. (That part is true—the rest I'm making up.) I want to build a court, and of course, bleachers around it. The bleachers will be the same depth (*by "depth" I mean the length from the court to the back of the bleachers) on all four sides. I want the total area of the room to be 8,000 square feet. How deep must the bleachers be?
3. Recall that the height of a ball thrown up into the air is given by the formula:

$$h(t) = h_o + v_o t - 16t^2$$

I am standing on the roof of my house, 20 feet up in the air. I throw a ball *up* with an initial velocity of 64 feet/sec. You are standing on the ground below me, with your hands 4 feet above the ground. The ball travels up, then falls down, and then you catch it. How long did it spend in the air?

Solve by completing the square

4. $x^2 + 6x + 8 = 0$

5. $x^2 - 10x + 30 = 5$

6. $x^2 + 8x + 20 = 0$

7. $x^2 + x = 0$

8. $3x^2 - 18x + 12 = 0$

9. Consider the equation $x^2 + 4x + 4 = c$ where c is some constant. *For what values of c will this equation have...*

a. Two real answers?

b. One real answer?

c. No real answers?

10. Solve by completing the square: $x^2 + 6x + a = 0$. (a is a constant.)

Name: _____

The “Generic” Quadratic Equation

OK, let’s say I wanted to solve a quadratic equation by completing the square. Here are the steps I would take, illustrated on an example problem. (These steps are exactly the same for any

Note that as I go along, I simplify things—for instance, rewriting $3\frac{1}{2} + 9$ as $12\frac{1}{2}$, or $\sqrt{12\frac{1}{2}}$ as $\frac{5}{\sqrt{2}}$. It is *always* a good idea to simplify as you go along!

Step

Example

The problem itself

$$2x^2 - 3x - 7 = 9x$$

Put all the x terms on one side, and the number on the other

$$2x^2 - 12x = 7$$

Divide both sides by the coefficient of x^2

$$x^2 - 6x = 3\frac{1}{2}$$

Add the same number to both sides. What number?

$$x^2 - 6x + 9 = 3\frac{1}{2} + 9$$

Half the coefficient of x , squared. (The coefficient of x is -6 .)Half of that is -3 . So we add 9 to both sides.)

Rewrite the left side as a perfect square

$$(x - 3)^2 = 12\frac{1}{2}$$

Square rootâ [U+0080] [U+0094] but with a â [U+0080] [U+009C] plus or minusâ [U+0080] [U+009D]!

(Remember, if x^2 is 25, x may be 5 or -5 !)

$$x - 3 = \pm \sqrt{12\frac{1}{2}} = \pm \sqrt{6\sqrt{2}}$$

Finally, add or subtract the number next to the x

$$x = 3 \pm \frac{5}{\sqrt{2}} (\approx -0.5, 6.5)$$

Now, you’re going to go through that same process, only you’re going to start with the “generic” quadratic equation:

$$ax^2 + bx + c = 0$$

As you know, once we solve this equation, we will have a formula that can be used to solve anyevery quadratic equation is just a specific case of that one!

Walk through each step. Remember to simplify things as you go along!

1. Put all the x terms on one side, and the number on the other.2. Divide both sides by the coefficient of x^2 .

3. Add the same number to both sides. What number? Half the coefficient of

- What is the coefficient of x ?
- What is $\frac{1}{2}$ of that?
- What is that squared?

OK, now add that to both sides of the equation.

- This brings us to a “rational expressions moment”—on the right side of the equation you will be adding two fractions. Go ahead and add them!
- Rewrite the left side as a perfect square.
- Square root—but with a “plus or minus”! (*Remember, if $x^2 = 25$, x may be 5 or -5 !)
- Finally, add or subtract the number next to the x .

Did you get the good old quadratic formula? If not, go back and see what’s wrong. If you did, give it a try on these problems! (Don’t solve these by factoring or completing the square, solve them using the quadratic formula that you just derived!)

8. $4x^2 + 5x + 1 = 0$

9. $9x^2 + 12x + 4 = 0$

10. $2x^2 + 2x + 1 = 0$

11. In general, a quadratic equation may have two real roots, *one real root*, or it may have *no real roots*. Based on the quadratic formula, and your experience with the previous three problems, how can you look at a quadratic equation $ax^2 + bx + c = 0$ and tell what kind of roots it will have?

Name: _____

Homework: Solving Quadratic Equations

- $2x^2 - 5x - 3 = 0$
 - Solve by factoring
 - Solve by completing the square
 - Solve by using the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
 - Which way was easiest? Which way was hardest?
 - Check your answers by plugging back into the original equation.

For problems

2. $x^2 - 5x + 30 = 5(x + 1)$

3. $3x^2 + 24x + 60 = 0$

4. $\frac{2}{3}x^2 + 8.5x = \pi x$

5. $x^2 - x = 0$

6. $9x^2 = 16$

7. Consider the equation $x^2 + 8x + c = 0$ where c is some constant. *For what values of c will this equation have...*

- Two real answers?
- One real answer?
- No real answers?

8. Starting with the generic quadratic equation $ax^2 + bx + c = 0$, complete the square and derive the quadratic formula. As much as possible, do this without consulting your notes.

Name: _____

Sample Test: Quadratic Equations I

1. Multiply:

a. $(x - \frac{3}{2})^2$

b. $(x + \sqrt{3})^2$

c. $(x - 7)(x + 7)$

d. $(x - 2)(x^2 - 4x + 4)$

e. $(x + 3)(2x - 5)$

f. Check your answer to part (e) by substituting in the number 1 for x into both the original expression, and your resultant expression. Do they come out the same? (*No credit* here for just saying, “Yes”—I have to be able to see your work!)

2. Here is a formula you probably never saw, but it is true: for any x and a , $(x + a)^4 = x^4 + 4x^3a + 6x^2a^2 + 4xa^3 + a^4$. Use that formula to expand the following.

a. $(x + 2)^4 =$

b. $(x - 1)^4 =$

3. Factor:

a. $x^2 - 36$

b. $2x^2y - 72y$

c. Check your answer to part (b) by multiplying back. (*I have to see your work!)

d. $x^3 - 6x^2 + 9x$

e. $3x^2 - 27x + 24$

f. $x^2 + 5x + 5$

g. $2x^2 + 5x + 2$

4. Geoff has a rectangular yard which is 55' by 75'. He is designing his yard as a big grassy rectangle, surrounded by a border of mulch and bushes. The border will be the same width all the way around. The area of his entire yard is 4125 square feet. The grassy area will have a smaller area, of course—Geoff needs it to come out exactly 3264 square feet. How wide is the mulch border?

5. Standing outside the school, David throws a ball up into the air. The ball leaves David's hand 4' above the ground, traveling at 30 feet/sec. Raven is looking out the window 10' above ground, bored by her class as usual, and sees the ball go by. How much time elapsed between when David threw the ball, and when Raven saw it go by? **To solve this problem, use the equation** $h(t) = h_o + v_ot - 16t^2$.

6. Solve by factoring: $2x^2 - 11x - 30 = 0$

7. Solve by completing the square: $2x^2 + 6x + 4 = 0$

8. Solve by using the quadratic formula: $-x^2 + 2x + 1 = 0$

9. Solve. No credit unless I see your work! $ax^2 + bx + c = 0$

Solve any way you want to

10. $2x^2 + 4x + 10 = 0$

11. $(\frac{1}{2})x^2 - x + 2\frac{1}{2} = 0$

12. $x^3 = x$

13. Consider the equation $3x^2 - bx + 2 = 0$, where b is some constant. *For what values of b will this equation have...*
- No real answers:
 - Exactly one answer:
 - Two real answers:
 - Can you find a value of b for which this equation will have two *rational* answers—that is, answers that can be expressed with no square root? (Unlike a-c, I'm not asking for *all* such solutions, just one.)

Extra Credit (5 points): *one negative answer and one positive answer.* Give your problem in words—then show the equation that represents your problem—then solve the equation—then answer the original problem in words.

Name: _____

Graphing Quadratic Functions

1. Graph by plotting points. Make sure to include positive and negative values of x ! $y = x^2$

Note that there is a little point at the bottom of the graph. This point is called the “vertex.”

Graph each of the following by drawing these as variations of

2. $y = x^2 + 3$ Vertex:

3. $y = x^2 - 3$ Vertex:

4. $y = (x - 5)^2$ Vertex:

5. Plot a few points to verify that your graph 4 is correct.

6. $y = (x + 5)^2$ Vertex:

7. $y = 2x^2$ Vertex:

8. $y = \frac{1}{2}x^2$ Vertex

9. $y = -x^2$ Vertex:

In these graphs, each problem transforms the graph in several

10. $y = (x - 5)^2 - 3$ Vertex:

11. Make a graph on the calculator to verify that your graph of 10 is correct.

12. $y = 2(x - 5)^2 - 3$ Vertex:

13. $y = -2(x - 5)^2 - 3$ Vertex:

14. $y = \frac{1}{2}(x + 5)^2 + 3$ Vertex:

15. Where is the vertex of the general graph $y = a(x - h)^2 + k$?

16. Graph by plotting points. Make sure to include positive and negative values of y ! $x = y^2$

x

y

Graph by drawing these as variations of $x = y^2$.

17. $x = y^2 + 4$

18. $x = (y - 2)^2$

19. Plot a few points to verify that your graph of 18 is correct.

20. $x = -y^2$

21. $x = -2(y - 2)^2 + 4$

Name: _____

Graphing Quadratic Functions II

Yesterday we played a bunch with quadratic functions, by seeing how they took the equation $y = x^2$ and permuted it. Today we're going to start by making some generalizations about all that.

1. $y = x^2$

- Where is the vertex?
- Which way does it open (up, down, left, or right?)
- Draw a quick sketch of the graph.

2. $y = 2(x - 5)^2 + 7$

- Where is the vertex?
- Which way does it open (up, down, left, or right?)
- Draw a quick sketch of the graph.

3. $y = (x + 3)^2 - 8$

- Where is the vertex?
- Which way does it open (up, down, left, or right?)
- Draw a quick sketch of the graph.

4. $y = -(x - 6)^2$

- Where is the vertex?
- Which way does it open (up, down, left, or right?)
- Draw a quick sketch of the graph.

5. $y = -x^2 + 10$

- Where is the vertex?
- Which way does it open (up, down, left, or right?)
- Draw a quick sketch of the graph.

6. Write a set of rules for looking at any quadratic function in the form $y = a(x - h)^2 + k$ and telling where the vertex is and which way it opens.

7. Now, all of those (as you probably noticed) were vertical/horizontal parabolas. Write a set of rules for looking at any quadratic function in the form $x = a(y - k)^2 + h$ and telling where the vertex is, and which way it opens.

After you complete 7, stop and let me check your rules before you go on any further.

OK, so far, so good! But you may have noticed a problem already, which is that most quadratic functions that we've dealt with in the past did not look like $y = a(x - h)^2 + k$. They looked more like... well, you know, $x^2 - 2x - 8$ or something like that. How do we graph that?

Answer: we put it into

OK, but how do we do that?

Answer: Completing the square! The process is almost *like*, and (more importantly) it *is not like*, the completing the square we did before!

Step

The function itself

We used to start by putting the number (-8 in this case) on the other side.

In this case, we don't have another side.

But I still want to set that -8 apart.

So I'm going to put the rest in parentheses that's where we're going to complete the square.

Inside the parentheses, add the number you need to complete the square.

Problem is, we used to add this number to both sides but as I said before,

We have no other side. So I'm going to add it inside the parentheses,

and at the same time subtract it outside the parentheses,

so the function is, in total, unchanged.

Inside the parentheses, you now have a perfect square and can rewrite it as such.

Outside the parentheses, you just have two numbers to combine.

Voila! You can now graph it!

Your turn!

8. $y = x^2 + 2x + 5$

a. Complete the square, using the process I used above, to make it $y = a(x - h)^2 + k$.

b. Find the vertex and the direction of opening, and draw a quick sketch.

9. $x = y^2 - 10y + 15$

a. Complete the square, using the process I used above, to make it $x = a(y - k)^2 + h$.

b. Find the vertex and the direction of opening, and draw a quick sketch.

Name: _____

Homework: Graphing Quadratic Functions II

Put each equation in the form $y = a(x - h)^2 + k$ or $x = a(y - k)^2 + h$, and graph.

a. $y = x^2$

b. $y = x^2 + 6x + 5$

c. Plot at least three points to verify your answer to 2.

d. $y = x^2 - 8x + 16$

e. $y = x^2 - 7$

f. $y + x^2 = 6x + 3$

g. Use a graph on the calculator to verify your answer to 6.

h. $y + x^2 = x^2 + 6x + 9$

i. $y = -2x^2 + 12x + 4$

j. $x = 3y^2 + 6y$

k. $x^2 + y^2 = 9$

l. Explain in words how you can look at any equation, in any form, and tell if it will graph as a parabola or not.

Name: _____

Example

$$x^2 - 6x - 8$$

$$(x^2 - 6x) - 8$$

$$(x^2 - 6x + 9) - 9 - 8$$

$$(x - 3)^2 - 17$$

Vertex $(3, -17)$

Solving Problems by Graphing Quadratic Functions

Let's start with our ball being thrown up into the air. As you doubtless recall:

$$h(t) = h_o + v_o t - 16t^2$$

1. A ball is thrown upward from the ground with an initial velocity of 64 ft/sec.
 - a. Write the equation of motion for the ball.
 - b. Put that equation into standard form for graphing.
 - c. Now draw the graph. h (the height, and also the dependent variable) should be on the y -axis, and t (the time, and also the independent variable) should be on the x -axis.
 - d. Use your graph to answer the following questions: at what time(s) is the ball on the ground?
 - e. At what time does the ball reach its maximum height?
 - f. What is that maximum height?
2. Another ball is thrown upward, this time from the roof, 30' above ground, with an initial velocity of 200 ft/sec.
 - a. Write the equation of motion for the ball.
 - b. Put that equation into standard form for graphing, and draw the graph as before.
 - c. At what time(s) is the ball on the ground?
 - d. At what time does the ball reach its maximum height?
 - e. What is that maximum height?

OK, we're done with the height equation for now. The following problem is taken from a Calculus book. Just so you know

3. A farmer has 2400 feet of fencing, and wants to fence off a rectangular field that borders a straight river. He needs no fence along the river. What are the dimensions of the field that has the largest area?
 - a. We're going to start by getting a "feeling" for this problem, by doing a few drawings. First of all, draw the river, and the fence around the field by the river, assuming that the farmer makes his field 2200 feet long. How far out from the river does the field go? What is the total area of the field?

After you do part (a), please stop and check with me, so we can make sure you have the right idea, before going on to part (b)

- b. Now, do another drawing where the farmer makes his field only 400 feet long. How far out from the river does the field go? What is the total area of the field?
- c. Now, do another drawing where the farmer makes his field 1000 feet long. How far out from the river does the field go? What is the total area of the field?

The purpose of all that was to make the point that if the field is

- d. Do a final drawing, but this time, label the length of the field simply x . How far out from the river does the field go?
- e. What is the area of the field, as a function of x ?
- f. Rewrite $A(x)$ in a form where you can graph it, and do a quick sketch. (Graph paper not necessary, but you do need to label the vertex.)
- g. Based on your graph, how long should the field be to maximize the area? What is that maximum area? (Hint: make sure the area comes out bigger than all the other three you already did, or something is wrong!)

Name: _____

Homework: Solving Problems by Graphing Quadratic Functions

Just as we did in class, we will start with our old friend

1. Michael Jordan jumps into the air at a spectacular 24 feet/second.
 - a. Write the equation of motion for the flying Wizard.
 - b. Put that equation into standard form for graphing, and draw the graph as before.
 - c. How long does it take him to get back to the ground?
 - d. At what time does His Airness reach his maximum height?
 - e. What is that maximum height?
2. Time to generalize! A ball is thrown upward from the ground with an initial velocity of v_0 . At what time does it reach its maximum height, and what is that maximum height?

Some more problems from my Calculus books

3. Find the dimensions of a rectangle with perimeter 100 ft whose area is as large as possible. (Of course this is similar to the one we did in class, but without the river.)
4. There are lots of pairs of numbers 10 : for instance, $8 + 2$, or $9\frac{1}{2} + \frac{1}{2}$. Find the two that have the largest product possible.
5. A pharmaceutical company makes a liquid form of penicillin. If they manufacture x units, they sell them for $200x$ dollars (in other words, they charge \$200 per unit). However, the total *cost* of manufacturing x units is $500,000 + 80x + 0.003x^2$. How many units should they manufacture to maximize their profits?

Name: _____

Quadratic Inequalities

The first part of this assignment is brought to you by our unit on

The following graph shows the temperature throughout the month of March. Actually, I just made this graph up—the numbers do not actually reflect the temperature throughout the month of March. We're just pretending, OK?

1. On what days was the temperature exactly 0°C ?
2. On what days was the temperature below freezing?
3. On what days was the temperature above freezing?
4. What is the domain of this graph?
5. During what time periods was the temperature going up?
6. During what time periods was the temperature going down?
7. The following graph represents the graph $y = f(x)$.
 - a. Is it a function? Why or why not?
 - b. What are the zeros?
 - c. For what x -values is it positive?
 - d. For what x -values is it negative?
 - e. Draw the graph $y = f(x) - 2$.

f. Draw the graph $y = -f(x)$.

OK, your memory is now officially refreshed, right? You remember how to look at a graph and see when it is zero, when it is below zero, and when it is above zero

Now we get to the actual “quadratic inequalities” part. But the good news is, there is nothing new here! First you will graph the function (you already know how to do that). Then you will identify the region(s) where the graph is positive, or negative (you already know how to do that).

8. $x^2 + 8x + 15 > 0$

a. Draw a quick sketch of the graph by finding the zeros, and noting whether the function opens up or down.

b. Now, the inequality asks when that function is > 0 — that is, when it is positive. Based on your graph, for what x -values is the function positive?

c. Based on your answer to part (b), choose one x -value for which the inequality *should* hold, and one for which it should *not*. Check to make sure they both do what they should.

9. A flying fish jumps from the surface of the water with an initial speed of 4 feet/sec.

a. Write the equation of motion for this fish.

b. Put it in the correct form, and graph it.

c. Based on your graph, answer the question: during what time interval was the fish above

d. During what time interval was the fish below

e. At what time(s) was the fish exactly at the level of the water?

f. What is the maximum height the fish reached in its jump?

Name: _____

Homework: Quadratic Inequalities

1. $x^2 + 8x + 7 > 2x + 3$

a. Put in standard form (with a zero on the right) and graph

b. Based on your graph, for what x -values is this inequality true?

c. Based on your answer, choose one x -value for which the inequality *should* hold, and one for which it should *not*. Check to make sure they both do what they should.

2. $2x^2 + 8x + 8 = 0$

a. Graph

b. Based on your graph, for what x -values is this inequality true?

c. Based on your answer, choose one x -value for which the inequality *should* hold, and one for which it should *not*. Check to make sure they both do what they should.

3. $-2x^2 + 8x > 9$

a. Put in standard form (with a zero on the right) and graph

b. Based on your graph, for what x -values is this inequality true?

c. Based on your answer, choose one x -value for which the inequality *should* hold, and one for which it should *not*. Check to make sure they both do what they should.

4. $-x^2 + 4x + 3 > 0$

a. Graph the function

b. Based on your graph, for what x -values is this inequality true?

c. Based on your answer, choose one x -value for which the inequality *should* hold, and one for which it should *not*. Check to make sure they both do what they should.

5. $x^2 > x$

a. Put in standard form (with a zero on the right) and graph

b. Based on your graph, for what x -values is this inequality true?

c. Now, let's solve the original equation a different way—divide both sides by x . Did you get the same answer this way? If not, which one is correct? (Answer by trying points.) What went wrong with the other one?

6. $x^2 + 6x + c < 0$

a. For what values of c will this inequality *be true in some range*?

b. For what values of c will this inequality *never* be true?

c. For what values of c will this inequality *always* be true?

Name: _____

Sample Test: Quadratics II

a. $x = -3y^2 + 5$

a. Opens (up / down / left / right)

b. Vertex: _____

c. Sketch a quick graph on the graph paper

b. $y + 9 = 2x^2 + 8x + 8$

a. Put into the standard form of a parabola.

b. Opens (up / down / left / right)

c. Vertex: _____

d. Sketch a quick graph on the graph paper

c. For what x -values is this inequality true? $2x^2 + 8x + 8 < 9$

d. For what x -values is this inequality true? $-x^2 - 10x \leq 28$

e. A rock is thrown up from an initial height of 4' with an initial velocity of 32 ft/sec. As I'm sure you recall, $h(t) = h_0 + v_0t - 16t^2$.

a. Write the equation of motion for the rock.

b. At what time (how many seconds after it is thrown) does the rock reach its peak? How high is that peak? (Don't forget to answer both questions...)

c. During what time period is the rock above ground?

f. A hot dog maker sells hot dogs for \$2 each. (So if he sells x hot dogs, his revenue is $2x$.) His cost for manufacturing x hot dogs is $100 + \frac{1}{2}x + \frac{1}{1000}x^2$. (Which I just made up.)

a. Profit *revenue minus cost*. Write a function $P(x)$ that gives the profit he will make, as a function of the number of hot dogs he makes and sells.

b. How many hot dogs should he make in order to maximize his profits? What is the maximum profit?

c. How many hot dogs does he need to make, in order to make any profit at all? (The answer will be in the form "as long as he makes more than this *less than that* or, in other words, *between this and that*, he will make a profit.")

Extra credit:

a) Find the vertex of the parabola $y = ax^2 + bx + c$. This will, of course, give you a generic formula for finding the vertex of any vertical parabola.

b) Use that formula $y = -3x^2 + 5x + 6$

CHAPTER **6**

Exponents

Chapter Outline

6.1 EXPONENTS

6.1 Exponents

Name: _____

Rules of Exponents

1. Here are the first six powers of two.

$$2^1 = 2$$

$$2^2 = 4$$

$$2^3 = 8$$

$$2^4 = 16$$

$$2^5 = 32$$

$$2^6 = 64$$

a. If I asked you for 2^7 (without a calculator), how would you get it? More generally, how do you always get *from one term in this list to the next term*? _____

b. Write an algebraic generalization to represent this rule. _____

2. Suppose I want to multiply 2^5 times 2^3 . Well, 2^5 means $2 \times 2 \times 2 \times 2 \times 2$, and 2^3 means $2 \times 2 \times 2$. So we can write the whole thing out like this.

$$\underbrace{2^5}_{2 * 2 * 2 * 2 * 2} * \underbrace{2^3}_{2 * 2 * 2} = \underbrace{2^8}_{2 * 2 * 2 * 2 * 2 * 2 * 2 * 2}$$

a. This shows that $(2^5)(2^3) = 2^{[]}$

b. Using a similar drawing, demonstrate what $(10^3)(10^4)$ must be.

c. Now, write an algebraic generalization for this rule. _____

d. Show how your answer to 1b (the “getting from one power of two, to the next in line”) is a special case of the more general rule you came up with in 2c (“multiplying two exponents”).

3. Now we turn our attention to division. What is $\frac{3^{12}}{3^{10}}$?

a. Write it out explicitly. (Like earlier I wrote out explicitly what $2^5 2^3$ was: expand the exponents into a big long fraction.)

b. Now, cancel all the like terms on the top and the bottom. (That is, divide the top and bottom by all the 3s they have in common.)

c. What you are left with is the answer. So fill this in: $\frac{3^{12}}{3^{10}} = 3^{[]}$.

d. Write a generalization that represents this rule.

e. Suppose we turn it upside-down. Now, we end up with some 3s on the *bottom*. Write it out explicitly and cancel 3s, as you did before:

$$\frac{3^{10}}{3^{12}} = \frac{\quad}{\quad} = \frac{1}{3^{[\]}}$$

f. Write a generalization for the rule in part (e). Be sure to mention when that

4. Use all those generalizations to simplify $\frac{x^3y^3x^7}{x^5y^5}$

5. Now we're going to raise exponents, to $(2^3)^4$? Well, 2^3 means $2 \times 2 \times 2$. And when you raise anything to the fourth power, you multiply it by itself, four times. So we'll multiply *that* by itself four times:

$$(2^3)^4 = (2 \times 2 \times 2) (2 \times 2 \times 2) (2 \times 2 \times 2) (2 \times 2 \times 2)$$

a. So, just counting 2s, $(2^3)^4 = 2^{[\]}$.

b. Expand out $(10^5)^3$ in a similar way, and show what power of 10 it equals.

c. Find the algebraic generalization that represents this rule.

Name: _____

Homework: Rules of Exponents

Memorize these:

$$\begin{aligned} x^a x^b &= x^{a+b} \\ \frac{x^a}{x^b} &= x^{a-b} \text{ or } \frac{1}{x^{b-a}} \\ (x^a)^b &= x^{ab} \end{aligned}$$

Simplify, using these rules

- $3^{10} \times 3^5$
- $\frac{3^{10}}{3^5}$
- $\frac{3^5}{3^{10}}$
- $(3^5)^{10}$
- $(3^{10})^5$
- $3^{10} + 3^5$
- $3^{10} - 3^5$
- $\frac{6x^3y^2z^5}{4x^{10}yz^2}$
- $\frac{6x^2y^3+3x^3y}{xy+x^2y}$
- $\frac{(3x^2y)^2+xy}{(xy)^3}$
- $(3x^2 + 4xy)^2$

Name: _____

Extending the Idea of Exponents

1. Complete the following table

TABLE 6.1:

| x | 3^x |
|-----|-------------------------------------|
| 4 | $3 \times 3 \times 3 \times 3 = 81$ |
| 3 | |
| 2 | |
| 1 | |

- In this table, every time you go to the next row, what happens to the left-hand number (x)?
- What happens to the right-hand number (3^x)?
- Now, let's assume that pattern continues, and fill in the next few rows.

TABLE 6.2:

| x | 3^x |
|-----|-------|
| 1 | |

- Based on this table, $3^0 =$
- $3^{-1} =$
- $3^{-2} =$
- What would you expect 3^{-4} to be?
- Now check 3^{-4} on your calculator. Did it come out the way you predicted?

Name: _____

Homework: Extending the Idea of Exponents

Answer the following questions

- $5^0 =$
- $5^{-2} =$
- $(-2)^{-2} =$
- $(-2)^{-3} =$
- $6^{-3} =$
- $x^0 =$
- $x^{-a} =$

In those last two problems, of course, you have created the

- Let's look at the problem $6^0 6^x$ two different ways.
 - What is 6^0 ? Based on that, what is $6^0 6^x$?
 - What do our rules of exponents $6^0 6^x$?
- Let's look at the problem $\frac{6^0}{6^x}$ two different ways.
 - What is 6^0 ? Based on that, what is $\frac{6^0}{6^x}$?
 - What do our rules of exponents $\frac{6^0}{6^x}$?
- Let's look at the problem $6^{-4} 6^3$ two different ways.
 - What does 6^{-4} mean? Based on that, what is $6^{-4} 6^3$?

b. What do our rules of exponents say about $6^{-4}6^3$?

11. What would you square x^{36} ?

Now let's solve a few equations

12. Solve for x : $3^{x+2} = 3^{8-x}$. (*Hint*: If the bases are the same, the exponents must be the same!)

13. Solve for x : $2^{4x-3} = 8^{x-2}$. (*Hint*: Start by rewriting 8 as 2^3 , then use the rules of exponents.)

14. Solve for x : $5^{(3x^2+13x+10)} = 25^{(x+2)}$. (No more hints this time, you're on your own.)

15. Solve for x : $(7^x)(7^{(x+2)}) = 1$

16. Solve for x : $(7^x)^{(x+2)} = 1$

Name: _____

Fractional Exponents

1. On the homework, we demonstrated the rule of negative exponents by building a table. Now, we're going to demonstrate it another way—by using the rules of exponents.

a. According to the rules of exponents, $\frac{7^3}{7^5} = 7^{[]}$.

b. But if you write it out and cancel the excess 7s, then $\frac{7^3}{7^5} = \underline{\hspace{2cm}}$.

c. Therefore, since $\frac{7^3}{7^5}$ can only be one thing, we conclude that these two things must be equal: write that equation!

2. Now, we're going to approach fractional rules of exponents, $(9^{\frac{1}{2}})^2 = \underline{\hspace{2cm}}$

3. So, what does that tell us about $9^{\frac{1}{2}}$? Well, it is *some number* that when you square it, you get (* same answer you gave for number 2). So therefore, $9^{\frac{1}{2}}$ itself must be:

4. Using the same logic, what is $16^{\frac{1}{2}}$?

5. What is $25^{\frac{1}{2}}$?

6. What is $x^{\frac{1}{2}}$?

7. Construct a similar argument to show that $8^{\frac{1}{3}} = 2$.

8. What is $27^{\frac{1}{3}}$?

9. What is $(-1)^{\frac{1}{3}}$?

10. What is $x^{\frac{1}{3}}$?

11. What would you expect $x^{\frac{1}{3}}$ to be?

12. What is $25^{-\frac{1}{2}}$? (You have to combine the rules for *negative* and *fractional* exponents here!)

13. Check your answer to 12 on your calculator. Did it come out the way you expected?

OK, we've done negative exponents, and fractional exponents—but always with a 1?

14. Using the rules of exponents, $(8^{\frac{1}{3}})^2 = 8^{[]}$.

So that gives us a rule! We know what

15. $8^{\frac{2}{3}} =$

16. Construct a similar argument to show what $16^{\frac{3}{4}}$ should be.

17. Check $16^{\frac{3}{4}}$ on your calculator. Did it come out the way you predicted?

Now let's combine 3.)

18. $8^{-\frac{1}{3}} =$

19. $8^{-\frac{2}{3}} =$

For these problems, just say what it means. (For instance, $3^{\frac{1}{2}}$ means $\sqrt{3}$, end of story.)

20. 10^{-4}

21. $2^{-\frac{3}{4}}$

22. $x^{\frac{a}{b}}$

23. $x^{-\frac{a}{b}}$

Name: _____

Homework: Fractional Exponents

We have come up with the following definitions

$$\begin{aligned}x^0 &= 1 \\x^{-a} &= \frac{1}{x^a} \\x^{\frac{a}{b}} &= \sqrt[b]{x^a}\end{aligned}$$

Let's get a bit of practice using these definitions

1. $100^{\frac{1}{2}} =$

2. $100^{-2} =$

3. $100^{-\frac{1}{2}} =$

4. $100^{\frac{3}{2}} =$

5. $100^{-\frac{3}{2}} =$

6. Check all of your answers above on your calculator. If any of them did not come out right, figure out what went wrong, and fix it!

7. Solve for x : $\frac{x^{\frac{3}{2}}}{x^{\frac{1}{2}}} = 17^{\frac{1}{2}}$

8. Solve for x : $x^{\frac{1}{2}} = 9$

9. Simplify: $\frac{x}{\sqrt{x}}$

10. Simplify: $\frac{x^{\frac{3}{2}} + \sqrt{x}}{x^{\frac{5}{2}} + \frac{1}{\sqrt{x}}}$ (Hint: Multiply the top and bottom by $x^{\frac{1}{2}}$.)

Now... remember x and the y and then solving for y . Find the inverse of each of the following functions. To do this, in some cases, you will have to rewrite the things. For instance, in 9, you will start by writing $y = x^{\frac{1}{2}}$. Switch the x and the y , and you get $x = y^{\frac{1}{2}}$. Now what? Well, remember what that means: it means $x = \sqrt{y}$. Once you've done that, you can solve for y , right?

11. x^3

a. Find the inverse function.

b. Test it.

12. x^{-2}

a. Find the inverse function.

b. Test it.

13. x^0

a. Find the inverse function.

b. Test it.

14. Can you find a generalization about the inverse function of an exponent?

15. Graph $y = 2^x$ by plotting points. Make sure to include both positive and negative x values.

16. Graph $y = 2 \times 2^x$ by doubling all the y -values in the graph of $y = 2^x$.

17. Graph $y = 2^{x+1}$ by taking the graph $y = 2^x$ and “shifting” it to the left by one.

18. Graph $y = \left(\frac{1}{2}\right)^x$ by plotting points. Make sure to include both positive and negative x values.

Name: _____

“Real Life” Exponential Curves

The Famous King Exponent Story

This is a famous ancient story that I am not making up, except that I am changing some of the details.

A man did a great service for the king. The king offered to reward the man every day for a month. So the man said: “Your Majesty, on the first day, I want only a penny. On the second day, I want twice that: 2 . On the third day, I want twice that much again: 4 pennies. On the fourth day, I want 8 pennies, and so on. On the thirtieth day, you will give me the last sum of money, and I will consider the debt paid off.”

The king thought he was getting a great deal. . . but was he? Before you do the math, take a guess: how much do you think the king will pay the man on the

Now, let’s do the math. For each day, indicate how much money (in pennies) the king paid the man. Do this without a calculator, it’s good practice and should be quick.

Day 1: 1 penny

Day 2: 2

Day 3: 4

Day 4: 8

Day 5: _____

Day 6: _____

Day 7: _____

Day 8: _____

Day 9: _____

Day 10: _____

Day 11: _____

Day 12: _____

Day 13: _____

Day 14: _____

Day 15: _____

Day 16: _____

Day 17: _____

Day 18: _____

Day 19: _____

Day 20: _____

Day 21: _____

Day 22: _____

Day 23: _____

Day 24: _____

Day 25: _____

Day 26: _____

Day 27: _____

Day 28: _____

Day 29: _____

Day 30: _____

How was your guess?

Now let's get mathematical. On the n^{th} day, how many pennies did the king give the man? _____

Use your calculator, and the formula you just wrote down, to answer the question: what did the king pay the man on the 30th day? _____ Does it match what you put under “Day 30” above? (If not, something's wrong somewhere—find it and fix it!)

Finally, do a graph of this function, where the “day” is on the x -axis and the “pennies” is on the y axis (so you are graphing pennies as a function of day). Obviously, your graph won't get past the fifth or sixth day or so, but try to get an idea for what the shape looks like.

Compound Interest

Here is a slightly more realistic situation. Your bank pays 6% interest, compounded annually. That means that after the first year, they add 6% to your money. After the second year, they add another 6% to the new total... and so on.

You start with \$1,000. Fill in the following table.

TABLE 6.3:

| <u>Year</u> | <u>The bank gives you this...</u> | <u>and you end up with this</u> |
|-------------|-----------------------------------|---------------------------------|
| 0 | 0 | \$1000 |
| 1 | \$60 | \$1060 |
| 2 | \$63.60 | |
| 3 | | |
| 4 | | |
| 5 | | |

Now, let's start generalizing. Suppose at the end of one year, you have x dollars. How much does the bank give you that year?

And when you add that, how much do you have at the end of the next year? (Simplify as much as possible.)

So, now you know what is happening to your money each year. So after year n , how much money do you have? Give me an equation.

Test that equation to see if it gives you the same result you gave above for the end of year 5.

Once again, graph that. The x -axis should be year. The y -axis should be the total amount of money you end up with after each year.

How is this graph like, and how is it unlike, the previous graph?

If you withdraw all your money after $\frac{1}{2}$ a year, how much money will the bank give you? (Use the equation you found above!)

If you withdraw all your money after $2\frac{1}{2}$ years, how much money will the bank give you?

Suppose that, instead of starting \$1,000, I just tell you that you *had* \$1,000 at year 0. How much money did you have five years before that (year -5)?

How many years will it take for your money to triple? That is to say, in what year will you have \$3,000?

Name: _____

Homework: “Real Life” Exponential Curves

Radioactive substances decay according to a “half-life.” The half-life is the period of time that it takes for half the substance to decay. For instance, if the half-life is 20 minutes, then every 20 minutes, half the remaining substance decays.

As you can see, this is the sort of exponential curve that goes down instead of up: at each step (or half-life) the total amount divides by 2; or, to put it another way, *multiplies by* $\frac{1}{2}$.

First “Radioactive Decay” Case

You have 1 gram of a substance with a half-life of 1 minute. Fill in the following table.

TABLE 6.4:

| <u>Time</u> | <u>Substance remaining</u> |
|-------------|----------------------------|
| 0 | 1 gram |
| 1 minute | $\frac{1}{2}$ gram |
| 2 minutes | |
| 3 minutes | |
| 4 minutes | |
| 5 minutes | |

After n minutes, how many grams are there? Give me an equation.

Use that equation 5 minutes, how many grams of substance are there? Does your answer agree with what you put under “5 minutes” above? (If not, something’s wrong somewhere—find it and fix it!)

How much substance will be left after $4\frac{1}{2}$ minutes?

How much substance will be left after half an hour?

How long will it be before only one one-millionth of a gram remains?

Finally, on the attached graph paper, do a graph of this function, where the “minute” is on the x -axis and the “amount of stuff left” is on the y -axis (so you are graphing grams as a function of minutes). Obviously, your graph won’t get past the fifth or sixth minute or so, but try to get an idea for what the shape looks like.

Second “Radioactive Decay” Case

Now, we’re going to do a more complicated example. Let’s say you start with 1000 grams of a substance, and its

half-life is 20 minutes; that is, every 20 minutes, half the substance disappears. Fill in the following chart.

TABLE 6.5:

| <u>Time</u> | <u>Half-Lives</u> | <u>Substance remaining</u> |
|-------------|-------------------|----------------------------|
| 0 | 0 | 1000 grams |
| 20 minutes | 1 | 500 grams |
| 40 minutes | | |
| 60 minutes | | |
| 80 minutes | | |
| 100 minutes | | |

After n half-lives, how many grams are there? Give me an equation.

After n half-lives, how many *minutes* have gone by? Give me an equation.

Now, let's look at that equation the other way. After t minutes (for instance, after 60 minutes, or 80 minutes, *etc*), how many half-lives have gone by? Give me an equation.

Now we need to put it all together. After t minutes, how many grams are there? This equation should take you directly from the first column to the third: for instance, it should turn 0 into 1000, and 20 into 500. (*Note: you can build this as a *composite function*, starting from two of your previous answers!)

Test that equation to see if it gives you the same result you gave above after 100 minutes.

Once again, graph that on graph paper. The x -axis should be minutes. The y -axis should be the total amount of substance. In the space below, answer the question: how is it like, and how is it unlike, the previous graph?

How much substance will be left after 70 minutes?

How much substance will be left after two hours? (*Not two minutes, two hours!)

How long will it be before only one gram of the original substance remains?

Finally, a bit more about compound interest

If you invest $\$A$ into a bank with $i\%$ interest compounded n times per year, after t years your bank account is worth an amount M given by:

$$M = A \left(1 + \frac{i}{n} \right)^{nt}$$

For instance, suppose you invest $\$1,000$ in a bank that gives 10% interest, compounded "semi-annually" (twice a year). So A , your initial investment, is $\$1,000$. i , the interest rate, is 10%, or 0.10. n , the number of times compounded per year, is 2. So after 30 years, you would have:

$$\$1,000 \left(1 + \frac{0.10}{2} \right)^{2 \times 30} = \$18,679. \text{ (Not bad for a } \$1,000 \text{ investment!)}$$

Now, suppose you invest $\$1.00$ in a bank that gives 100% interest (nice bank!). How much do you have after one year if the interest is...

- Compounded annually (once per year)?
- Compounded quarterly (four times per year)?
- Compounded daily?
- Compounded every second?

Name: _____

Sample Test: Exponents

Simplify. Your answer should not contain any negative or fractional exponents

1. x^{-6}

2. x^0

3. $x^{\frac{1}{8}}$

4. $x^{\frac{2}{3}}$

5. $\left(\frac{2}{3}\right)^2$

6. $\left(\frac{1}{2}\right)^{-x}$

7. $(-2)^2$

8. $(-2)^3$

9. $(-2)^{-1}$

10. $(-9)^{\frac{1}{2}}$

11. $(-8)^{\frac{1}{3}}$

12. $y^{\frac{1}{4}}y^{\frac{3}{4}}$

13. $\frac{4x^4y^5z}{6wxy^2z^3}$

14. $\left(x^{\frac{1}{3}}\right)^3$

15. $\left(x^{\frac{1}{3}}\right)^2$

16. $x^{\frac{6}{3}}$

17. $x^{-\frac{3}{4}}$

18. $(4 \times 9)^{\frac{1}{2}}$

19. $4^{\frac{1}{2}} \times 9^{\frac{1}{2}}$

20. Give an algebraic formula that gives the generalization for 18 – 19.

Solve for x .

21. $8^x = 64$

22. $8^x = 8$

23. $8^x = 1$

24. $8^x = 2$

25. $8^x = \frac{1}{8}$

26. $8^x = \frac{1}{64}$

27. $8^x = \frac{1}{2}$

28. $8^x = 0$

29. Rewrite $\frac{1}{\sqrt[3]{x^2}}$ as X

something .

Solve for x .

30. $2^{(x+3)}2^{(x+4)} = 2$

31. $3^{(x^2)} = \left(\frac{1}{9}\right)^{3x}$

32. A friend of yours is arguing that $x^{\frac{1}{3}}$ should be defined to mean something to do with “fractions, or division, or something.” You say, “No, it means _____ instead.” He says, “That’s a crazy definition!” Give him a convincing argument *why* it should mean what you said it means.

33. On October 1st, I place 3 sheets of paper on the ground. Each day thereafter, I count the number of sheets on the ground, and add that many again. (So if there are 5 sheets, I add 5 more.) After I add my last pile on Halloween (October 31

st), how many sheets are there total?

- Give me the answer as a formula.
- Plug that formula into your calculator to get a number.
- If one sheet of paper is $\frac{1}{250}$ inches thick, how thick is the final pile?

34. **Depreciation.** The Web site www.bankrate.com defines *depreciation* as “the decline in a car’s value over the course of its useful life” (and also as “something new-car buyers dread”). The site goes on to say:

Let’s start with some basics. Here’s a standard rule of thumb about used cars. A car loses 15 percent to 20 percent of its value each year.

For the purposes of this problem, let’s suppose you buy a new car for exactly \$10,000, and it loses only 15% of its value every year.

- How much is your car worth after the first year?
- How much is your car worth after the second year?
- How much is your car worth after the n^{th} year?
- How much is your car worth after ten years? (This helps you understand why new-car buyers dread depreciation.)

35. Draw a graph of $y = 2 \times 3^x$. Make sure to include negative and positive values of x .

36. Draw a graph of $y = \left(\frac{1}{3}\right)^x - 3$. Make sure to include negative and positive values of x .

37. What are the domain and range of the function you graphed in number 36?

Extra credit: $(a + b)^2$ is not, in general, the same as $a^2 + b^2$. But under what circumstances, if any, *are* they the same?

CHAPTER **7**

Logarithms

Chapter Outline

7.1 LOGARITHMS

7.1 Logarithms

Name: _____

Introduction to Logarithms

- a. On day 0, you have 1 penny. Every day, you double.
 - a. How many pennies do you have on day 10?
 - b. How many pennies do you have on day n ?
 - c. On what day 32 pennies? Before you answer, *express this question as an equation, where x is the variable you want to solve for.*
 - d. Now, what is x ?
- b. A radioactive substance is decaying. There is currently 100g of the substance.
 - a. How much substance will there be after 3 half-lives?
 - b. How much substance will there be after n half-lives?
 - c. After how many half-lives 1g of the substance left? Before you answer, *express this question as an equation, where x is the variable you want to solve for.*
 - d. Now, what is x ? (Your answer will be approximate.)

In both of the problems above, part (d) required you to invert *from time to amount*, it asked you to go *from amount to time*. (This is what an inverse function does—it goes the other way—remember?)

So let's go ahead and talk formally about an inverse exponential function. Remember that an inverse function goes backward $f(x) = 2^x$ turns a 3 into an 8, then $f^{-1}(x)$ must turn an 8 into a 3.

So, fill in the following table (on the left) with a bunch of x and y values for the mysterious *inverse function of 2^x* . Pick x -values that will make for easy y -values. See if you can find a few x -values that make y be 0 or negative numbers!

On the right, fill in x and y values for the *inverse function of 10^x* .

TABLE 7.1:

| Inverse of 2^x | Inverse of 2^x |
|------------------|------------------|
| x | y |
| 8 | 3 |

TABLE 7.2:

| Inverse of 10^x | Inverse of 10^x |
|-------------------|-------------------|
| x | y |

Now, let's see if we can get a bit of a handle on this type of function.

In some ways, it's like a square root. \sqrt{x} is the inverse of x^2 . When you see \sqrt{x} you are really seeing a mathematical question: "What number, squared, gives me x ?"

Now, we have the inverse of 2^x (which is quite different from x^2 of course). But this new function is also a question: see if you can figure out what it is. That is, try to write a question that will reliably get me from the left-hand column to the right-hand column in the first table above.

Do the same for the second table above.

Now, come up with a word problem of your own, similar to the first two in this exercise, but related to compound interest.

Name: _____

Homework—Logs

$\log_2 8 =$ asks the question: “2 to what power is 8?” Based on that, you can answer the following questions:

1. $\log_2 8 =$
2. $\log_3 9 =$
3. $\log_{10} 10 =$
4. $\log_{10} 100 =$
5. $\log_{10} 1000 =$
6. $\log_{10} 1,000,000 =$
7. Looking at your answers to problems 3 – 6, what does the \log_{10} tell you about a number?
8. Multiple choice: which of the following is closest to $\log_{10} 500$?
 - A. 1
 - B. $1\frac{1}{2}$
 - C. 2
 - D. $2\frac{1}{2}$
 - E. 3
9. $\log_{10} 1 =$
10. $\log_{10} \frac{1}{10} =$
11. $\log_{10} \frac{1}{100} =$
12. $\log_{10}(0.01) =$
13. $\log_{10} 0 =$
14. $\log_{10}(-1) =$
15. $\log_9 81 =$
16. $\log_9 \frac{1}{9} =$
17. $\log_9 3 =$
18. $\log_9 \frac{1}{81} =$
19. $\log_9 \frac{1}{3} =$
20. $\log_5(5^4) =$
21. $5^{\log_5 4} =$

OK. When I say, $\sqrt{36} = 6$, that’s the *same thing* as saying $6^2 = 36$. Why? Because $\sqrt{36}$ asks a question: “What squared equals 36?” So the equation $\sqrt{36} = 6$ is providing an answer: “six squared equals 36.”

You can look at logs in a similar way. If I say $\log_2 32 = 5$ I’m asking a question: “2 to what power is 32?” And I’m

answering: “5. 2 to the *fifth* power is 32.” So saying $\log_2 32 = 5$ is the *same thing* as saying $2^5 = 32$.

Based on this kind of reasoning, rewrite the following logarithm statements as exponent statements

22. $\log_2 8 = 3$

23. $\log_3\left(\frac{1}{3}\right) = -1$

24. $\log_x(1) = 0$

25. $\log_a x = y$

Now do the same thing backward: rewrite the following exponent statements as logarithm statements

26. $4^3 = 64$

27. $8^{-\frac{2}{3}} = \frac{1}{4}$

28. $a^b = c$

Finally... you don't understand a function until you graph it

29. a. Draw a graph of $y = \log_2 x$. Plot at least 5 points to draw the graph.

b. What are the domain and range of the graph? What does that tell you about this function?

Name: _____

Properties of Logarithms

a. $\log_2(2) =$

b. $\log_2(2 \times 2) =$

c. $\log_2(2 \times 2 \times 2) =$

d. $\log_2(2 \times 2 \times 2 \times 2) =$

e. $\log_2(2 \times 2 \times 2 \times 2 \times 2) =$

f. Based on numbers 1 – 5, finish this sentence in words: when you take \log_2 of a number, you find:

g. $\log_2(8) =$

h. $\log_2(16) =$

i. $\log_2(8 \times 16) =$

j. $\log_3(9) =$

k. $\log_3(27) =$

l. $\log_3(9 \times 27) =$

m. Based on numbers 7 – 12, write an algebraic generalization about logs.

n. Now, let's dig more deeply into that one. Rewrite problems 7 – 9 so they look like problems 1 – 5: that is, so the thing you are taking the log of is written as a power of 2.

a. 7:

b. 8:

c. 9:

d. Based on this rewriting, can you explain why 13 works?

o. $\log_5(25) =$

p. $\log_5\left(\frac{1}{25}\right) =$

q. $\log_2(32) =$

r. $\log_2\left(\frac{1}{32}\right) =$

s. Based on numbers 15 – 18, write an algebraic generalization about logs.

t. $\log_3(81) =$

u. $\log_3(81 \times 81) =$

v. $\log_3(81)^2 =$

w. $\log_3(81 \times 81 \times 81) =$

x. $\log_3(81)^3 =$

y. $\log_3(81 \times 81 \times 81 \times 81) =$

z. $\log_3(81)^4 =$

. Based on numbers 20 – 26 write an algebraic generalization about logs.

Name: _____

Homework—Properties of Logarithms

Memorise these three rules

$$\log_x(ab) = \log_x a + \log_x b$$

$$\log_x\left(\frac{a}{b}\right) = \log_x a - \log_x b$$

$$\log_x(a^b) = b \log_x a$$

1. In class, we demonstrated the *first* *third* rules above. For instance, for the first rule:

$$\log_2 8 = \log_2(2 \times 2 \times 2) = 3$$

$$\log_2 16 = \log_2(2 \times 2 \times 2 \times 2) = 4$$

$$\log_2(8 \times 16) = \log_2[(2 \times 2 \times 2)(2 \times 2 \times 2 \times 2)] = 7$$

This demonstrates that when you multiply two numbers, their logs add

Now, you come up with a similar demonstration of the second rule of logs, that shows why when you divide *subtract*.

Now we're going to practice applying *Take my word for these two facts*. (You don't have to memorize them, but you will be using them for this homework.)

- $\log_5 8 = 1.29$

- $\log_5 60 = 2.54$

Now, use those facts to answer the following questions

2. $\log_5 480 =$

(Hint: $480 = 8 \times 60$. So this is $\log_5(8 \times 60)$. Which rule above helps you rewrite this?)

3. How can you use your calculator to test your answer to 2? (I'm assuming here that you can't find $\log_5 480$ on your calculator, but you can do exponents.) Run the test—did it work?

4. $\log_5\left(\frac{2}{15}\right) =$

5. $\log_5\left(\frac{15}{2}\right) =$

6. $\log_5 64 =$

7. $\log_5(5)^{23} =$

8. $5(\log_5 23) =$

Simplify, using the

9. $\log_a(x \cdot x \cdot x \cdot x)$

10. $\log_a(x \cdot 1)$

Below bracket are different size

Simplify, using the $\log\left(\frac{x}{y}\right)$ property:

11. $\log_a\left(\frac{1}{x}\right)$

12. $\log_a\left(\frac{x}{1}\right)$

13. $\log_a\left(\frac{x}{x}\right)$

Simplify, using the

14. $\log_a(x)^4$

15. $\log_a(x)^0$

16. $\log_a(x)^{-1}$

17. a. Draw a graph of $y = \log_{\frac{1}{2}} x$. Plot at least 5 points to draw the graph.

b. What are the domain and range of the graph? What does that tell you about this function?

Name: _____

Using the Laws of Logarithms

$$\log_x(ab) = \log_x a + \log_x b$$

$$\log_x\left(\frac{a}{b}\right) = \log_x a - \log_x b$$

$$\log_x(a^b) = b \log_x a$$

1. Simplify: $\log_3(x^2) - \log_3(x)$

2. Simplify: $\log_3(9x) - \log_3(x)$

3. Simplify: $\frac{\log(x^2)}{\log(x)}$

4. Solve for x .

$$\log(2x + 5) = \log(8 - x)$$

5. Solve for x :

$$\log(3) + \log(x + 2) = \log(12)$$

6. Solve for x :

$$\ln(x) + \ln(x - 5) = \ln(14)$$

7. Solve for y in terms of x :

$$\log(x) = \log(5y) - \log(3y - 7)$$

So What Are Logarithms Good For, Anyway?

1. **Compound Interest.** Andy invests \$1,000 in a bank that pays out 7% interest, compounded annually. Note that your answers to parts (a) and (c) will be numbers, but your answers to parts (b) and (d) will be formulae.

- After 3 years, how much money does Andy have?
- After t years, how much money m does Andy have? $m(t) =$
- After how many years does Andy have exactly \$14,198.57?
- After how many years t does Andy have $\$m$? $t(m) =$

2. **Sound Intensity.** Sound is a wave in the air—the loudness of the sound is related to the intensity of the wave. The intensity of a whisper is approximately 100; the intensity of a normal conversation is approximately 1,000,000. Assuming that a person starts whispering at time $t = 0$, and gradually raises his voice to a normal conversational level by time $t = 10$, show a possible graph of the intensity of his voice. (*You can't get the graph exactly, since you only know the beginning and the end, but show the general shape.)

3. That was pretty complicated, wasn't it? It's almost impossible to graph or visualize something going from a hundred to a million: the range is too big.

Fortunately, sound volume is usually not measured in intensity, but in loudness, $L = 10 \log_{10} I$, where L is the loudness (measured in decibels), and I is the intensity.

- What is the loudness, in decibels, of a whisper?
- What is the loudness, in decibels, of a normal conversation?
- Now do the graph again—showing an evolution from whisper to conversation in 30 seconds—but this time, graph loudness instead of intensity.
- That was a heck of a lot nicer, wasn't it? (This one is sort of rhetorical.)
- The quietest sound a human being can hear is intensity 1. What is the loudness of that sound?
- The sound of a jet engine—which is roughly when things get so loud they are painful—is loudness 120 decibels. What is the intensity of that sound?
- The formula I gave above gives loudness as a function of intensity. Write the opposite function, that gives intensity as a function of loudness.
- If sound A is *twenty decibels higher* than sound B , *how much more intense* is it?

4. **Earthquake intensity.** When an Earthquake occurs, seismic detectors register the shaking of the ground, and are able to measure the “amplitude” (fancy word for “how big they are”) of the waves. However, just like sound intensity, this amplitude varies so much that it is very difficult to graph or work with. So Earthquakes are measured on the *Richter scale* which is the \log_{10} of the amplitude ($r = \log_{10} a$).

- A “microearthquake” is defined as 2.0 or less on the Richter scale. Microearthquakes are not felt by people, and are detectable only by local seismic detectors. If a is the amplitude of an earthquake, write an *inequality* that must be true for it to be classified as a microearthquake.
- A “great earthquake” has amplitude of 100,000,000 or more. There is generally one great earthquake somewhere in the world each year. If r is the measurement of an earthquake on the Richter scale, write an *inequality* that must be true for it to be classified as a great earthquake.
- Imagine trying to show, on a graph, the amplitudes of a bunch of earthquakes, ranging from microearthquakes to great earthquakes. (Go on, just imagine it—I'm not going to make you do it.) A lot easier with the Richter scale, ain't it?
- Two Earthquakes are measured—the second one has 1000 times the amplitude of the first. What is the difference in their measurements on the Richter scale?

5. pH. In Chemistry, a very important quantity is the *concentration of Hydrogen ions*, written as $[H^+]$ —this is related to the acidity of a liquid. In a normal pond, the concentration of Hydrogen ions is around 10^{-6} moles/liter. (In other words, every liter of water has about 10^{-6} , or $\frac{1}{1,000,000}$ moles of Hydrogen ions.) Now, acid rain begins to fall on that pond, and the concentration of Hydrogen ions begins to go up, until the concentration is 10^{-4} moles/liter (every liter has $\frac{1}{10,000}$ moles of H^+).

a. How much did the concentration go up by?

b. Acidity is usually not measured as concentration (because the numbers are very unmanageable, as you can see), but as pH, which is defined as $-\log_{10}[H^+]$. What is the *pH* of the normal pond?

c. What is the pH of the pond after the acid rain?

6. Based on numbers 2 – 5, write a brief description of *what kind of function* generally leads scientists to want to use a logarithmic scale.

Name: _____

Homework: What Are Logarithms Good For, Anyway?

1. I invest \$300 in a bank that pays 5% interest, compounded annually. So after t years, I have $300(1.05)^t$ dollars in the bank. When I come back, I find that my account is worth \$1000. How many years has it been? *Your answer will not be a number—it will be a formula with a log in it.*

2. The pH of a substance is given by the formula $\text{pH} = -\log_{10}[H^+]$, where $[H^+]$ is the concentration of Hydrogen ions.

a. If the Hydrogen concentration is $\frac{1}{10,000}$, what is the pH?

b. If the Hydrogen concentration is $\frac{1}{1,000,000}$, what is the pH?

c. What happens to the pH every time the Hydrogen concentration *divides* by 10?

You may have noticed that all our logarithmic functions use the base 10. Because this is so common, it is given a special name: the *common log*. When you see something like $\log(x)$ with no base written at all, that means the log is 10. (So $\log(x)$ is a shorthand way of writing $\log_{10}(x)$, just like \sqrt{x} is a shorthand way of writing $\sqrt[2]{x}$. With roots, if you don't see a little number there, you assume a 2. With logs, you assume a 10.)

3. In the space below, write the question that $\log(x)$ asks.

Use the common log to answer the following questions

4. $\log 100$

5. $\log 1,000$

6. $\log 10,000$

7. \log (1 with n 0s after it)

8. $\log 500$ (use the log button on your calculator)

OK, so the log button on your calculator does common logs, that is, logs base 10.

There is one other log button on your calculator. It is called the “natural log,” and it is written \ln (which sort of stands for “natural log” only backward—personally, I blame the French).

\ln means the log to the base e . What is e ? It's a long ugly number—kind of like π only different—it goes on forever and you can only approximate it, but it is somewhere around 2.7. Answer the following questions about the natural log.

9. $\ln(e) =$

10. $\ln(1) =$

11. $\ln(0) =$

12. $\ln(e^5) =$

13. $\ln(3) =$ (*this is the only one that requires the ln button on your calculator)

Name: _____

Sample Test: Logarithms

1. $\log_3 3 =$

2. $\log_3 9 =$

3. $\log_3 27 =$

4. $\log_3 30 =$

(approximately)

5. $\log_3 1 =$

6. $\log_3\left(\frac{1}{3}\right) =$

7. $\log_3\left(\frac{1}{9}\right) =$

8. $\log_3(-3) =$

9. $\log_9 3 =$

10. $3^{\log_3 8} =$

11. $\log_{-3} 9 =$

12. $\log 100,000 =$

13. $\log \frac{1}{100,000} =$

14. $\ln e^3 =$

15. $\ln 4 =$

16. Rewrite as a logarithm equation (no exponents): $q^z = p$

17. Rewrite as an exponent equation (no logs): $\log_w g = j$

For questions

$$\log_5 12 = 1.544$$

$$\log_5 20 = 1.861$$

18. $\log_5 240 =$

19. $\log_5\left(\frac{3}{5}\right) =$

20. $\log_5 1\frac{2}{3} =$

21. $\log_5\left(\frac{3}{5}\right)^2 =$

22. $\log_5 400 =$

23. Graph $y = -\log_2 x + 2$.

24. What are the domain and range of the graph you drew in 23?

25. I invest \$200 in a bank that pays 4% interest, compounded annually. So after y years, I have $200(1.04)^y$ dollars in the bank. When I come back, I find that my account is worth \$1000. How many years has it been? *Your answer will be a formula with a log in it.*26. The "loudness" of a sound is given by the formula $L = 10\log I$, where L is the loudness (measured in decibels),

and I is the intensity of the sound wave.

- If the sound wave intensity is 10, what is the loudness?
 - If the sound wave intensity is 10,000, what is the loudness?
 - If the sound wave intensity is 10,000,000, what is the loudness?
 - What happens to the loudness every time the sound wave intensity multiplies 1,000?
27. Solve for x .

$$\ln(3) + \ln(x) = \ln(21)$$

28. solve for x .

$$\log_2(x) + \log_2(x + 10) = \log_2(11)$$

Extra credit: $x. e^x = (\mathbf{cabin})$

CHAPTER

8

Rational Expressions

Chapter Outline

8.1 RATIONAL EXPRESSIONS

8.1 Rational Expressions

Name: _____

Rational Expressions

a. $\frac{1}{x} + \frac{1}{y}$

a. Add

b. Check your answer by plugging $x = 2$ and $y = 4$ into both my original expression, and your simplified expression. Do not use calculators or decimals.

b. $\frac{1}{x} - \frac{1}{y} =$

c. $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right) =$

d. $\frac{\frac{1}{y}}{\frac{1}{x}} =$

e. $\frac{x^2+2x+1}{x^2-1}$

a. Simplify

b. Check your answer by plugging $x = 3$ into both my original expression, and your simplified expression. Do not use calculators or decimals.c. Are there any x -values for which the new expression does not equal the old expression?

f. $\frac{2}{x^2-9} - \frac{4}{x^2+2x-15} =$

g. $\frac{4x^2-25}{x^2+2x+1} \times \frac{x^2+4x+3}{2x^2+x-15}$

a. Multiply

b. Check your answer by plugging $x = -2$ into both my original expression, and your simplified expression. (If they don't come out the same, you did something wrong!)c. Are there any x -values for which the new expression does *not* equal the old expression?

Name: _____

Homework—Rational Expressions

a. $\frac{x^2-6x+5}{x^2-16} \times \frac{x^2+8x+16}{x^2-7x+10}$

a. Simplify

b. What values of x are not allowed in the original expression?c. What values of x are not allowed in your simplified expression?

b. $\frac{\frac{2x^2+7x+3}{x^3+4x}}{\frac{x^2+x-6}{3x}}$

a. Simplify

b. What values of x are not allowed in the original expression?c. What values of x are not allowed in your simplified expression?

c. $\frac{1}{x-1} - \frac{1}{x+1}$

a. Simplify

b. What values of x are not allowed in the original expression?c. What values of x are not allowed in your simplified expression?d. Test your answer by choosing an x value and plugging it into the original expression, and your simplified expression. Do they yield the same answer?

- d. $\frac{x-3}{x^2+9x+20} - \frac{x-4}{x^2+8x+15}$
- Simplify
 - What values of x are not allowed in the original expression?
 - What values of x are not allowed in your simplified expression?
- e. $\frac{x+1}{4x^2-9} + \frac{4x}{6x^2-9x}$
- Simplify
 - Test your answer by choosing an x value and plugging it into the original expression, and your simplified expression. Do they yield the same answer?

Name: _____

Rational Equations

1. Suppose I tell you that $\frac{x}{36} = \frac{15}{36}$. What are all the values x can take that make this statement true?

OK, that was easy, wasn't it? So the moral of that story is: rational equations are easy to solve, if they have a common denominator. If they don't, of course, you just get one!

2. Now suppose I tell you that $\frac{x}{18} = \frac{15}{36}$. What are all the values x can take that make this statement true?

Hey,

Umm, yeah

OK, one more pretty easy one

3. $\frac{x^2+2}{21} = \frac{9}{7}$

Did you get only one answer? Then look again—this one has two!

Once you are that far, you've got the general idea—get a common denominator, and then set the numerators equal. So let's really get into it now...

4. $\frac{x+2}{x+3} = \frac{x+5}{x+4}$

5. $\frac{2x+6}{x+3} = \frac{x+5}{2x+7}$

6. $\frac{x+3}{2x-3} = \frac{x-5}{x-4}$

- Solve. You should end up with two answers.
- Check both answers by plugging them into the original equation.

Name: _____

Homework: Rational Expressions and Equations

- a. $\frac{1}{x-1} - \frac{1}{2x}$
- Simplify
 - What values of x are not allowed in the original expression?
 - What values of x are not allowed in your simplified expression?
 - Test your answer by choosing an x value and plugging it into the original expression, and your simplified expression. Do they yield the same answer?
- b. $\frac{3x^2+5x-8}{6x+16} \times \frac{4x^3+x}{3x^3-3x}$
- Simplify
 - What values of x are not allowed in the original expression?

- c. What values of x are not allowed in your simplified expression?
- c. $\frac{2}{4x^3-x} + \frac{3}{2x^3+3x^2+x}$
- Simplify
 - What values of x are not allowed in the original expression?
 - What values of x are not allowed in your simplified expression?
- d. $\frac{\frac{x^2+9}{x^2-4}}{\frac{x^2+7x+12}{x^2+2x-8}}$
- Simplify
 - What values of x are not allowed in the original expression?
 - What values of x are not allowed in your simplified expression?
 - Test your answer by choosing an x value and plugging it into the original expression, and your simplified expression. Do they yield the same answer?
- e. $\frac{x}{x^2-25} + \langle \text{something} \rangle = \frac{x^2+1}{x^3-9x^2+20x}$
- What is the something?
 - What values of x are not allowed in the original expression?
 - What values of x are not allowed in your simplified expression?
- f. $\frac{x-6}{x-3} = \frac{x+18}{2x+7}$
- Solve for x . You should get two answers.
 - Check by plugging one

Name: _____

Dividing Polynomials

- a. $\frac{28k^3p-42kp^2+56kp^3}{14kp} =$
- b. $\frac{x^2-12x-45}{x+3} =$
- $\frac{2y^2+y-16}{y-3} =$
 - Test your answer by choosing a number for y and seeing if you get the same answer.
- c. $\frac{2h^3-5h^2+22h+51}{2h+3} =$
- d. $\frac{2x^3-4x^2+6x-15}{x^2+3} =$
- e. $\frac{x^3-4x^2}{x-4} =$
- $\frac{x^3-27}{x-3} =$
 - Test your answer by multiplying back.
- f. After dividing two polynomials, I get the answer $r^2 - 6r + 9 - \frac{1}{r-3}$. What two polynomials did I divide?

Name: _____

Sample Test: Rational Expressions

- a. $\frac{x-3}{x^2+9x+20} - \frac{x-4}{x^2+8x+15}$
- Simplify
 - What values of x are not allowed in the original expression?
 - What values of x are not allowed in your simplified expression?

- b. $\frac{2}{x^2-1} + \frac{x}{x^2-2x+1}$
- Simplify
 - What values of x are not allowed in the original expression?
 - What values of x are not allowed in your simplified expression?
- c. $\frac{4x^3-9x}{x^2-3x-10} \times \frac{2x^2-20x+50}{6x^2-9x}$
- Simplify
 - What values of x are not allowed in the original expression?
 - What values of x are not allowed in your simplified expression?
- d. $\frac{\frac{1}{x}}{\frac{x-1}{x^2}}$
- Simplify
 - What values of x are not allowed in the original expression?
 - What values of x are not allowed in your simplified expression?
- e. $\frac{x-1}{2x-1} = \frac{x+7}{7x+4}$
- Solve for x .
 - Test one of your answers and show that it works in the original expression. (No credit unless you show your work!)
- f. $\frac{6x^3-5x^2-5x+34}{2x+3}$
- Solve by long division.
 - Check your answer (show your work!!!).

Extra credit: x and y , that have this curious property: their sum is the same as their *product*. (Sum means “add them”; product means “multiply them.”)

- Can you find any such pairs?
- To generalize: if one of my numbers is x , can you find a general formula that will always give me the other one?
- Is there any number x that has no possible y to work with?

CHAPTER **9**

Radicals

Chapter Outline

9.1 RADICALS

9.1 Radicals

Name: _____

Radicals (aka* Roots)

As a student of mine once said, “In real life, no one ever says, ‘Here’s 100 dollars, what’s the square root of it?’” She’s right, of course—as far as I know, no one *ever* takes the square root of money. And she is asking exactly the right question, which is: why do we need roots anyway?

1. If a square is 49 ft^2 in area, how long are the sides?

Q. You call that real life?

A. OK, you asked for it. . .

2. A real estate developer is putting houses down on a plot of land that is 50 acres large. He wants to put down 100 houses, so each house will sit on a $\frac{1}{2}$ -acre lot. (1 acre is 43,560 square feet.) If each house sits on a square lot, how long are the sides of each lot?

3. A piano is dropped from a building 100 ft high. (“Dropped” implies that someone just let go of it, instead of throwing it—so it has no initial velocity.)

a. Write the equation of motion for this piano, recalling as always that

$$h(t) = h_o + v_o t - 16t^2.$$

b. According to the equation, how high is the piano when $t = 0$? Explain in words what this answer means.

c. After 2 seconds, how high is the piano?

d. How many seconds does it take the piano to reach the ground?

e. Find the inverse function t when the piano reaches any given height h .

Convinced? Square roots come up all the time in real life, because squaring things

Radicals and Exponents

If I tell you that $\sqrt{25} = 5$, that is the *same thing* as telling you that $5^2 = 25$. Based on that kind of logic, rewrite the following radical equations (4–6) as exponent equations.

4. $\sqrt{100} = 10$

5. $\sqrt[3]{8} = 2$

6. $\sqrt[a]{b} = c$

7. Now, rewrite all three as logarithm*You mean we still have to know *that*?)

Some Very Important Generalizations

8. $\sqrt{9} =$

9. $\sqrt{4} =$

10. $\sqrt{9} \times \sqrt{4} =$

11. $\sqrt{9 \times 4} =$

12. Based on 8-11, write an algebraic generalization.

13. Now, give me a completely different example of that same generalization: four different statements, like 8-11,

that could be used to generate that same generalization.

14. $\frac{\sqrt{9}}{\sqrt{4}} =$

15. $\sqrt{\frac{9}{4}}$

a. What do you think is the answer?

b. Test by squaring back. (To square anything, multiply it by itself. So this just requires multiplying fractions!) If it doesn't work, try something else, until you are convinced that you have a good $\sqrt{\frac{9}{4}}$.

16. Based on numbers 14-15, write an algebraic generalization.

17. $\sqrt{9} + \sqrt{4} =$

18. $\sqrt{9+4} =$

19. Based on numbers 17-18, write an algebraic generalization that is not

Simplifying Radicals

Based on the generalization you wrote in 12, and given the fact that $4 \times 2 = 8$, we can simplify $\sqrt{8}$ as follows.

$$\sqrt{8} = \sqrt{4 \times 2} \text{ (because 8 is the same thing as } 4 \times 2 \text{)}$$

$$\sqrt{4 \times 2} = \sqrt{4} \times \sqrt{2} \text{ (this is where we use that generalization)}$$

$$\sqrt{4} \times \sqrt{2} = 2\sqrt{2} \text{ because } \sqrt{4} \text{ is the same thing as } 2 \text{)}$$

So, we see that $\sqrt{8}$ is the same thing as $2\sqrt{2}$.

20. Find $\sqrt{8}$ on your calculator.

21. Find $\sqrt{2}$ on your calculator.

22. Double your answer to 21 on your calculator.

23. So, did it work?

24. Good! Then let's try another one. Simplify $\sqrt{72}$, using the same steps I took to simplify $\sqrt{8}$.

25. Check your answer on the calculator. Did it work?

26. Oh yeah, one final question: what is $\sqrt{x^{16}}$? How can you test your answer?

Name: _____

Homework—Radicals

For the following problems, I am

1. $\sqrt{64}$

2. $\sqrt[3]{64}$

3. $3\sqrt{64}$

4. $-\sqrt{64}$

5. $\sqrt{-64}$

6. $\sqrt[3]{-64}$

7. If $\sqrt[n]{-x}$ has a real answer, what can you say about n ?

8. $\sqrt{8}$

9. $\sqrt{18}$

10. $\sqrt{48}$

11. $\sqrt{70}$

12. $\sqrt{72}$

13. $\sqrt{100}$

14. $\sqrt{\frac{1}{4}}$

15. $\sqrt{\frac{4}{9}}$

16. $\sqrt{\frac{8}{12}}$

17. $\sqrt{16x^2}$

a. Simplify as much as possible (just like all the other problems)

b. Check your answer with $x = 3$. Did it work?

18. $\sqrt{(16x)^2}$

19. $\sqrt{x^{10}}$

20. $\sqrt{x^{11}}$

21. $\sqrt{75x^3y^6z^5}$

22. $\sqrt{x^2y^2}$

23. $\sqrt{x^2 + y^2}$

24. $\sqrt{x^2 + 2xy + y^2}$

25. $\sqrt{x^2 + 9}$

26. $\sqrt{x^2 + 6x + 9}$ Say, remember inverse functions?

27. $f(x) = x^2$.

a. Find the inverse function.

b. Test it.

28. $f(x) = x^3$.

a. Find the inverse function.

b. Test it.

29. $f(x) = 3^x$.

a. Find the inverse function.

b. Test it.

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A Bunch of Other Stuff About Radicals

Let's start off with a bit of real life again, shall we?

1. Albert Einstein's "Special Theory of Relativity" tells us that matter and energy are different forms of the same thing. $E = mc^2$, where E is the amount of energy, m is the amount of matter, and c is the speed of light. So, suppose I did an experiment where I converted m kilograms of matter, and wound up with E Joules of energy. Give me the equation I could use that would help me figure out, from these two numbers, what the speed of light is.

2. The following figure is an Aerobie, or a washer, or whatever you want to call it—it's the shaded area, a ring with inner thickness r_1 and outer thickness r_2 .

- What is the area of this shaded region, in terms of r_1 and r_2 ?
- Suppose I told you that the area of the shaded region is 32π , and that the inner radius r_1 is 7. What is the outer radius r_2 ?
- Suppose I told you that the area of the shaded region is A , and that the outer radius is r_2 . Find a formula for the inner radius.

OK, that's enough about real life. Let's try simplifying a few expressions, using the rules we developed yesterday.

3. $\sqrt{100y}$

4. $\sqrt{\frac{x}{25}}$

5. $\sqrt{x+16}$

6. $\frac{\sqrt{50}}{2} =$

7. $5\sqrt{2} + 2\sqrt{3} - 3\sqrt{2} =$

8. $\sqrt{27} - \sqrt{48} =$

Let's try some that are a bit trickier—sort of like rational expressions. Don't forget to start by getting a common denominator!

9. $\frac{1}{\sqrt{2}} + \frac{\sqrt{2}}{2} =$

a. Simplify. (Don't use your calculator, it won't help.)

b. Now, check

10. $\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} =$

a. Simplify. (Don't use your calculator, it won't help.)

b. Now, check

11. $\frac{1}{\sqrt{3}+1} - \frac{\sqrt{3}}{2} =$

a. Simplify. (Don't use your calculator, it won't help.)

b. Now, check

And now, the question you knew I would ask...

12. Graph $y = \sqrt{x}$.

- Plot a whole mess of points. (Choose x -values that will give you pretty easy-to-graph y -values!)
- What is the domain? What is the range?
- Draw the graph.

13. Graph $y = \sqrt{x} - 3$ by shifting the previous graph.

- Plug in a couple of points to make sure your "shift" was correct. Fix it if it wasn't.
- What is the domain? What is the range?

14. Graph $y = \sqrt{x-3}$ by shifting the previous graph.

- Plug in a couple of points to make sure your "shift" was correct. Fix it if it wasn't.
- What is the domain? What is the range?

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Homework: A Bunch of Other Stuff About Radicals

- Several hundred years before Einstein, Isaac Newton proposed a theory of gravity. According to Newton's theory, any two bodies exert a force on each other, pulling them closer together. The force is given by the equation $F = G \frac{m_1 m_2}{r^2}$, where F is the force of attraction, G is a constant, m_1 and m_2 are the masses of the two different bodies, and r is the distance between them. Find a formula that would give you r if you already knew F, G, m_1 , and m_2 .
- In the following drawing, m is the height (vertical height, straight up) of the mountain; s is the length of the ski lift (the diagonal line); and x is the horizontal distance from the bottom of the ski lift to the bottom of the mountain.
 - Label these three numbers on the diagram. Note that they make a right triangle.
 - Write the relationship between the three. (Pythagorean Theorem)
 - If you build the ski lift starting 1,200 feet from the bottom of the mountain, and the mountain is 800 feet high, how long is the ski lift?
 - If the ski lift is s feet long, and you build it starting x feet from the bottom of the mountain, how high is the mountain?
- Simplify $\frac{\sqrt{28}}{4-\sqrt{7}}$. Check your answer on your calculator.
- Simplify $\frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}-2}$. Check your answer on your calculator.
- Graph $y = \sqrt{x} - 3$. What is the domain and range?
- Graph $y = \sqrt{x-3}$. What is the domain and range?

Name: _____

Radical Equations

Before I get into the radical equations, there is something very important I have to get out of the way. Square these two out:

1. $(2 + \sqrt{2})^2 =$

2. $(\sqrt{3} + \sqrt{2})^2 =$

How'd it go? If you got six for the first answer and five for the second, stop!

Now, radical equations. Let's start off with an easy radical equation.

3. $\sqrt{2}x + 3 = 7$

I call this an "easy" radical equation because there is no x under the square root. Sure, there's a $\sqrt{2}$, but that's just a number. So you can solve it pretty much the same way you would solve $4x + 3 = 7$; just subtract 3, then divide by $\sqrt{2}$.

a. Solve for x .

b. Check your answer by plugging it into the original equation. Does it work?

This next one is definitely trickier, but it is still in the category that I call "easy" because there is still no x under the square root.

4. $\sqrt{2}x + 3x = 7$

a. Solve for x .

b. Check your answer by plugging it into the original equation. Does it work? (Feel free to use your calculator, but show me what you did and how it came out.)

Now, what if there is x under the square root? Let's try a basic one like that.

5. Solve for x : $\sqrt{x} = 9$

What did you get? If you said the answer is three: shame, shame. The square root of 3 isn't 9, is it? Try again.

OK, that's better. You probably guessed your way to the answer. But if you had to be systematic about it, you could say, "I got to the answer by squaring both sides." The rule is: whenever there is an

It worked out this time, but squaring both sides is fraught with peril. Here are a few examples.

6. $\sqrt{x} = -9$

a. Solve for x , by squaring both sides.

b. Check your answer by plugging it into the original equation.

Hey, what happened? When you square both sides, you get $x = 81$, just like before. But this time, it's the wrong answer: $\sqrt{81}$ is *not* 9. The moral of the story is that *when you square both sides, you can introduce false answers*. So whenever you square both sides, you have to check your answers to see if they work. (We will see that rule come up again in some much less obvious places, so it's a good idea to get it under your belt now: *whenever you square both sides, you can introduce false answers!*)

But that isn't the only danger of squaring both sides. Check this out. . .

7. Solve for $2 + \sqrt{x} = 5$

Hey, what happened there? When you square the left side, you got (I hope) $x + 4\sqrt{x} + 4$. Life isn't any simpler, is it? So the lesson there is, *you have to get the square root by itself before you can square both sides*. Let's come back to that problem.

8. $2 + \sqrt{x} = 5$

a. Solve for x by first getting the square root by itself, and then squaring both sides

b. Check your answer in the original equation.

Whew! Much better! Some of you may have never fallen into the trap—you may have just subtracted the two to begin with. But you will find you need the same technique for harder problems, such as this one:

9. $x - \sqrt{x} = 6$

a. Solve for x by first getting the square root by itself, and then squaring both sides, and then solving the resulting equation. (*Note: you should end up with two answers.*)

b. Check your answers in the original equation. (*Note: if you did everything right, you should find that one answer works and the other doesn't. Once again, we see that squaring both sides can introduce false answers!*)

The more times you see x under a square root, the more squaring you have to do. For instance. . .

10. $\sqrt{x-2} = \sqrt{3} - \sqrt{x}$

What do you do now? You're going to have to square both sides. . . that will simplify the left, but the right will still be complicated. But if you look closely, you will see that you have changed an equation with x under the square root *twice*, into an equation with x under the square root *once*. So then, you can solve it the way you did above: get the square root by itself and square both sides. Before you are done, you will have squared both sides twice!

Solve

Name: _____

Homework: Radical Equations

For each of the following, you will first identify it as one of three types of problem:

- No x under a radical, so don't square both sides.
- x under a radical, so you will have to isolate it and square both sides.
- More than one x under a radical, so you will have to isolate-and-square more than once!

Then you will solve it; and finally, you will check your answers (often on a calculator). Remember that if you squared both sides, you may get false answers even if you did the problem correctly! If you did not square both sides, a false answer means you must have made a mistake somewhere.

- a. $\sqrt{x} = [U+0080] [U+0093]3$
 - a. Which type of problem is it?
 - b. Solve for x .
 - c. Check your answer(s).
- b. $\sqrt{2x} - 3 = \sqrt{3x}$
 - a. Which type of problem is it?
 - b. Solve for x .
 - c. Check your answer(s).
- c. $x - \sqrt{2x} = 4$
 - a. Which type of problem is it?
 - b. Solve for x .
 - c. Check your answer(s).
- d. $\sqrt{4x+2} - \sqrt{2x} = 1$
 - a. Which type of problem is it?
 - b. Solve for x .
 - c. Check your answer(s).
- e. $3 - \sqrt{x-2} = -4$
 - a. Which type of problem is it?
 - b. Solve for x .
 - c. Check your answer(s).
- f. $x + 2\sqrt{x} = 15$
 - a. Which type of problem is it?
 - b. Solve for x .
 - c. Check your answer(s).
- g. $\sqrt{x+4} + \sqrt{x} = 2$
 - a. Which type of problem is it?
 - b. Solve for x .
 - c. Check your answer(s).

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Sample Test: Radicals

1. Punch this into your calculator and give the answer rounded to three decimal places. This is the only

$$\sqrt{69} =$$

2. Give me an approximate $\sqrt[3]{10}$

Simplify. Give answers using radicals, not decimals or approximations.

3. $\sqrt{400}$

4. $\sqrt[3]{-27}$

5. $3\sqrt{-27}$

6. $\sqrt{108}$

7. $\sqrt{20} - \sqrt{45}$

8. $\sqrt{\frac{300}{64}}$

9. $\sqrt{x^{16}}$

10. $\sqrt[5]{x^{38}}$

11. $\sqrt{98x^{20}y^5z}$

12. $\sqrt{4x^2 + 9y^4}$

13. $(\sqrt{3} - \sqrt{2})^2$

14. $\frac{\sqrt{24}}{2 + \sqrt{2}}$

15. $\sqrt[3]{\text{something}} = x + 2$. What is the *something*?

16. Rewrite as an exponent equation: $x = \sqrt[n]{y}$

17. Rewrite as a radical equation: $a^b = c$

18. Rewrite as a logarithm $a^b = c$

Solve for

19. $\frac{3 + \sqrt{x}}{2} = 5$

20. $\sqrt{2x + 1} - \sqrt{x} = 1$

21. $3x + \sqrt{3}(x) - 4 + \sqrt{8} = 0$

22. $\sqrt{x} + \sqrt{2}(x) - \sqrt{2} = 0$

23. For an object moving in a circle around the origin, whenever it is at the point (x, y) , its distance to the center of the circle is given by: $r = \sqrt{x^2 + y^2}$.

a. Solve this equation for x .

b. If $y = 2$ and $r = 2\frac{1}{2}$, what is x ?

24. Graph $y = [U+0080] [U+0093] \sqrt{x} + 3$.

25. What are the domain and range of the graph you drew in 24?

Extra credit: $y = \sqrt[3]{x}$.

CHAPTER **10**

Imaginary Numbers

Chapter Outline

10.1 IMAGINARY NUMBERS

10.1 Imaginary Numbers

Name: _____

Imaginary Numbers

1. Explain, using words and equations, why the equation $x^2 = -1$ has no answer, but $x^3 = -1$ does.

OK, so now we are going to use our imaginations. (Didn't think we were allowed to do that in math class, did you?) Suppose there were $x^2 = -1$? Obviously it wouldn't be a number that we are familiar with (such as 5, $-\frac{3}{4}$, or π). So, let's just give it a new name: i , because it's *imaginary*. What would it be like?

The definition of the imaginary number i is that it is the square root of -1 :

$$i = \sqrt{-1} \text{ or, equivalently, } i^2 = -1$$

Based on that definition, answer the following questions. In each case, don't just guess—give a good mathematical reason why the answer should be what you say it is!

2. What is $i(-i)$? (*Remember that $-i$ means $-1 \times i$.)

3. What is $(-i)^2$?

4. What is $(3i)^2$?

5. What is $(-3i)^2$?

6. What is $(\sqrt{2}i)^2$?

7. What is $(\sqrt{2}i)^2$?

8. What is $\sqrt{-25}$?

9. What is $\sqrt{-3}$?

10. What is $\sqrt{-8}$?

11. Fill in the following table.

$$\text{amp}; i^1$$

$$\text{amp}; i^2$$

$$\text{amp}; i^3$$

$$\text{amp}; i^4$$

$$\text{amp}; i^5$$

$$\text{amp}; i^6$$

$$\text{amp}; i^7$$

$$\text{amp}; i^8$$

$$\text{amp}; i^9$$

$$\text{amp}; i^{10}$$

$$\text{amp}; i^{11}$$

$$\text{amp}; i^{12}$$

12. Fill in the following table.

$$1^{100}$$

$$\text{amp}; i^{101}$$

$$\text{amp}; i^{102}$$

$$\text{amp}; i^{103}$$

$$\text{amp}; i^{104}$$

Now let's have some more fun!

13. $(3 + 4i)^2 =$

14. $(3 + 4i)(3 - 4i) =$

15. $\left(\frac{1}{i}\right)^2 =$

16. Simplify the fraction $\frac{1}{i}$. (*Hint: Multiply the top and bottom by i .*)

17. Square your answer to 16. Did you get the same answer you got to 15? Why or why not?

18. Simplify the fraction $\frac{1}{3+2i}$. (*Hint: Multiply the top and bottom by $3 - 2i$.*)

Name: _____

Homework: Imaginary Numbers

We began our in-class assignment by talking about why $x^3 = -1$ *does* have a solution, whereas $x^2 = -1$ does *not*. Let's talk about the same thing *graphically*.

1. On the graph below, do a quick sketch of $y = x^3$.

a. Draw, on your graph, all the points on the curve where $y = 1$. How many are there?

b. Draw, on your graph, all the points on the curve where $y = 0$. How many are there?

c. Draw, on your graph, all the points on the curve where $y = -1$. How many are there?

2. On the graph below, do a quick sketch of $y = x^2$.

a. Draw, on your graph, all the points on the curve where $y = 1$. How many are there?

b. Draw, on your graph, all the points on the curve where $y = 0$. How many are there?

c. Draw, on your graph, all the points on the curve where $y = -1$. How many are there?

3. Based on your sketch in number 2...

a. If a is some number such that $a > 0$, how many solutions are there to the equation $x^2 = a$?

b. If a is some number such that $a = 0$, how many solutions are there to the equation $x^2 = a$?

c. If a is some number such that $a < 0$, how many solutions are there to the equation $x^2 = a$?

d. If i is defined by the equation $i^2 = -1$, where the heck is it on the graph?

OK, let's get a bit more practice with i .

4. In class, we made a table of powers of i , and found that there was a repeating pattern.

Make that table again quickly below, to see the pattern.

i^1
 i^2
 i^3
 i^4
 i^5
 i^6
 i^7
 i^8
 i^9
 i^{10}
 i^{11}
 i^{12}

5. Now let's walk that table backward.

i^{-4}
 i^{-3}
 i^{-2}
 i^{-1}
 i^0

6. Did it work? Let's figure it out. What should i^0 be, according to our general rules of exponents?
7. What should i^{-1} be, according to our general rules of exponents? Can you simplify it to look like the answer in your table?
8. What should i^{-2} be, according to our general rules of exponents? Can you simplify it to look like the answer in your table?
9. What should i^{-3} be, according to our general rules of exponents? Can you simplify it to look like the answer in your table?
10. Simplify the fraction $\frac{i}{4-3i}$.

Name: _____

Complex Numbers

A complex number $a + bi$ where a and b are real numbers. a is the “real part” and bi is the “imaginary part.”

Examples are: $3 + 4i$ (a is 3, b is 4) and $3 - 4i$ (a is 3, b is -4).

- Is 4 a complex number? If so, what are a and b ? If not, why not?
- Is i a complex number? If so, what are a and b ? If not, why not?
- Is 0 a complex number? If so, what are a and b ? If not, why not?

All four operations—addition, subtraction, multiplication, and division—can be done to complex numbers, and the answer is always another complex number. So, for the following problems, let $X = 3 + 4i$ and $Y = 5 - 12i$. In each case, your answer should be a complex number, in the form $a + bi$.

4. Add: $X + Y$

5. Subtract: $X - Y$

6. Multiply: XY

7. Divide: $\frac{X}{Y}$ (*To get the answer in $a + bi$ form, you will need to use a trick we learned yesterday.)

8. Square: X^2

The complex conjugate($a + bi$) is defined as ($a - bi$). That is, the real part stays the same, and the imaginary part switches sign.

9. What is the complex conjugate of $(5 - 12i)$?

10. What do you get when you multiply $(5 - 12i)$ by its complex conjugate?

11. Where have we used complex conjugates before?

For two complex numbers to be equal, $2 + 3i$ is only equal to $2 + 3i$. It is not equal to $2 - 3i$ or to $3 + 2i$ or to anything else. So it is very easy to see if two complex numbers are the same, as long as they are both written in $a + bi$ form: you just set the *real parts equal*, and the *imaginary parts equal*. (If they are not written in that form, it can be very tricky to tell: for instance, we saw earlier that $\frac{1}{i}$ is the same as $-i$!)

12. If $2 - 3i = m + ni$, what are m and n ?

13. Solve for x and y : $(x - 6y) + (x + 2y)i = 1 - 3i$

Finally, remember... rational expressions? We can have some of those with complex numbers as well!

14. Simplify. As always, your answer should be in the form $a + bi$. $\frac{\left(\frac{4+2i}{3+2i}\right)}{\left(\frac{5-3i}{7-i}\right)}$

15. Simplify. $\frac{4+2i}{3+2i} - \frac{5-3i}{7-i}$

Name: _____

Homework: Complex Numbers

- $(3 + 7i) - (4 + 7i) =$
- $(5 - 3i) + (5 - 3i) =$
- $2(5 - 3i) =$
- $(5 - 3i)(2 + 0i) =$
- What is the complex conjugate of $(5 - 3i)$?
- What do you get when you multiply $(5 - 3i)$ by its complex conjugate?
- What is the complex conjugate of 7 ?
- What do you get when you multiply 7 by its complex conjugate?
- What is the complex conjugate of $2i$?
- What do you get when you multiply $2i$ by its complex conjugate?
- What is the complex conjugate of $(a + bi)$?
- What do you get when you multiply $(a + bi)$ by its complex conjugate?
- I'm thinking of a complex number z . When I multiply it by its complex conjugate (designated as z^*) the answer is 25.
 - What might z be?
 - Test it, and make sure it works—that is, that $(z)(z^*) = 25$!
- I'm thinking of a different z . When I multiply it by its complex conjugate, the answer is $3 + 2i$.
 - What might z be?

- b. Test it, and make sure it works—that is, that $(z)(z^*) = 3 + 2i!$
- o. Solve for x and y : $x^2 + 2x^2i + 4y + 40yi = 7 - 2i$
- p. Finally, a bit more exercise with rational expressions. We're going to take one problem and solve it two different ways. The problem is $\frac{3}{2+i} - \frac{7i}{3+4i}$. The final answer, of course, must be in the form $a + bi$.
- Here is one way to solve it: the common denominator is $(2 + i)(3 + 4i)$. Put both fractions over the common denominator and combine them. Then, take the resulting fraction, and simplify it into $a + bi$ form.
 - Here is a completely different way to solve the same problem. Take the two fractions we are subtracting and simplify them both $a + bi$ form, and then subtract.
 - Did you get the same answer? (If not, something went wrong...) Which way was easier?

Name: _____

Me, Myself, and the Square Root of i

We have already seen how to take a number such as $\frac{3}{2-i}$ and rewrite it in $a + bi$ format. There are many other numbers—such as 2^i and $\log(i)$ —that do not *look* like $a + bi$, but all of them can be *turned into* $a + bi$ form. In this assignment, we are going to find \sqrt{i} —that is, we are going to rewrite \sqrt{i} so that we can clearly see its real part and its imaginary part.

How do we do that? Well, we want to find some number z such that $z^2 = i$. And we want to express z in terms of its real and imaginary parts—that is, in the form $a + bi$. So what we want to solve is the following equation:

$$(a + bi)^2 = i$$

You are going to solve that equation now. When you find a and b , you will have found the answers.

Stop now and make sure you understand how I have set up this problem, before you go on to solve it.

- What is $(a + bi)^2$? Multiply it out.
- Now, rearrange your answer so that you have collected all the real terms together and all the imaginary terms together.

Now, we are trying to solve the equation $(a + bi)^2 = i$. So take the formula you just generated in number 2, and set it equal to i . This will give you two equations: one where you set the real part on the left equal to the real part on the right, and one where you set the imaginary part on the left equal to the imaginary part on the right.

- Write down both equations.
- Solve the two equations for a and b . (Back to “simultaneous equations,” remember?) In the end, you should have two (a, b) pairs that work in both equations.
- So... now that you know a and b , write down *two* complex answers to the problem $x^2 = i$.
- Did all that work? Well, let's find out. Take your answers in 5 and test them: that is, square it, and see if you get i . If you don't, something went wrong!
- OK, we're done! Did you get it all? Let's find out. Using a very similar technique to the one that we used here, find $\sqrt{-i}$: that is, find the two different solutions to the problem $z^2 = i$. Check them!

Name: _____

The Many Merry Cube Roots of -1

When you work with real numbers, $x^2 = 1$ has two different solutions (1 and -1). But $x^2 = -1$ has no solutions at all. When you allow for *complex* numbers, things are much more consistent: $x^2 = -1$ has two solutions, just like $x^2 = 1$. In fact, $x^2 = n$ will have two solutions for any number n —positive or negative, real or imaginary or complex. There is only one exception to this rule.

1. What is the one exception?

You might suspect that $x^3 = n$ should have *three* solutions in general—and you would be right! Let's take an example. We know that when we are working with real numbers, $x^3 = -1$ has only one solution.

2. What is the one solution?

But if we allow for complex $x^3 = -1$ has three possible solutions. We are going to find the other two.

How do we do that? Well, we know that every complex number can be written as $(a + bi)$, where a and b are real numbers. So if there is some complex number that solves $x^3 = -1$, then we can find it by solving the a and b that will make the following equation true:

$$(a + bi)^3 = -1$$

You are going to solve that equation now. When you find a and b , you will have found the answers.

Stop now and make sure you understand how I have set up this problem, before you go on to solve it.

3. What is $(a + bi)^3$? Multiply it out.

4. Now, rearrange your answer so that you have collected all the real terms together and all the imaginary terms together.

Now, we are trying to solve the equation $(a + bi)^3 = -1$. So take the formula you just generated in number 4, and set it equal to -1 . This will give you two equations: one where you set the real part on the left equal to the real part on the right, and one where you set the imaginary part on the left equal to the imaginary part on the right.

5. Write down both equations.

OK. If you did everything right, one of your two equations factors as $b(3a^2 - b^2) = 0$. If one of your two equations doesn't factor that way, go back—something went wrong!

If it did, then let's move on from there. As you know, we now have two things being multiplied to give 0, which means one of them must be 0. One possibility is that $b = 0$: we'll chase that down later. The other possibility is that $3a^2 - b^2 = 0$, which means $3a^2 = b^2$.

6. Solve the two equations for a and b by substituting $3a^2 = b^2$ into the other equation

7. So... now that you know a and b , write down *two* complex answers to the problem $x^3 = -1$. If you don't have two answers, look again!

8. But wait... shouldn't there be a third answer? Oh, yeah... what about that $b = 0$ business? Time to pick that one up. If $b = 0$, what is a ? Based on this a and b , what is the third and final solution to $x^3 = -1$?

9. Did all that work? Well, let's find out. Take either of your answers in 7 and test it: that is, cube it, and see if you get -1 . If you don't, something went wrong!

Name: _____

Homework: Quadratic Equations and Complex Numbers

I'm sure you remember the quadratic formula: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. Back when we were doing quadratic equations, if we wound up with a negative number under that square root, we just gave up. But now we can solve these equations!

- Use the quadratic formula to solve: $2x^2 + 6x + 5 = 0$.
- Use the quadratic formula to solve: $x^2 - 2x + 5 = 0$.
- Check one of your answers to 2.
- Solve by completing the square: $2x^2 + 10x + 17 = 0$.
- In general, what has to be true for a quadratic equation to have two complex roots?
- What is the relationship between the two complex roots?

g. Is it possible to have a quadratic equation with one

Name: _____

Sample Test: Complex Numbers

1. Fill in the following table.

$$\text{amp}; i^{-1}$$

$$\text{amp}; i^0$$

$$\text{amp}; i^1$$

$$\text{amp}; i^2$$

$$\text{amp}; i^3$$

$$\text{amp}; i^4$$

$$\text{amp}; i^5$$

$$\text{amp}; i^6$$

$$\text{amp}; i^7$$

Simplify.

2. $(i)^{85} =$

3. $(5i)^2 =$

4. $(ni)^{103} =$

5. a. $\sqrt{-20} =$

b. Other than *other* number that you can square to get -20 ? If so, what is it?

6. $(3a - bi)^2 =$

7. a. Complex conjugate of $4 + i =$

b. What do you get when you multiply $4 + i$ by its complex conjugate?

If the following are simplified to the form $a + bi$, what are a and b in each case?

8. $-i$

a. $a =$

b. $b =$

9. $\frac{n}{i}$

a. $a =$

b. $b =$

10. $\frac{5x}{1+2i} - \frac{2i}{3-i}$

a. $a =$

b. $b =$

11. If $2x + 3xi + 2y = 28 + 9i$, what are x and y ?

12. Make up a quadratic equation (using all real numbers) that has two complex roots, and solve it.

13. a. Find the two complex numbers (of course in the form $z = a + bi$) that fill the condition $z^2 = -2i$.

b. Check one of your answers to part (a), by squaring it to make sure you get $-2i$.

Extra credit: 2– dimensional graph: you graph the point $x + iy$ at (x, y) .

a. If you graph the point $5 + 12i$, how far is that point from the origin $(0, 0)$?

b. If you graph the point $x + iy$, how far is that point from the origin $(0, 0)$?

c. What do you get if you multiply the point $x + iy$ by its complex conjugate? How does this relate to your answer to part (b)?

CHAPTER **11**

Matrices

Chapter Outline

11.1 MATRICES

11.1 Matrices

Name: _____

Introduction to Matrices

The following matrix, stolen from a rusted lockbox in the back of a large, dark lecture hall in a school called Hogwarts, is the gradebook for Professor Severus Snape's class in potions.

TABLE 11.1:

| | Poison | Cure | Love philter | Invulnerability |
|----------------------|--------|------|--------------|-----------------|
| Granger, <i>H</i> | 100 | 105 | 99 | 100 |
| Longbottom, <i>N</i> | 80 | 90 | 85 | 85 |
| Malfoy, <i>D</i> | 95 | 90 | 0 | 85 |
| Potter, <i>H</i> | 70 | 75 | 70 | 75 |
| Weasley, <i>R</i> | 85 | 90 | 95 | 90 |

When I say this is a “matrix” I’m referring to the numbers in boxes. The labels (such as “Granger, *H*” or “Poison”) are labels that help you understand the numbers in the matrix, but they are not the matrix itself.

Each student is designated by a row.

1. Below, copy the row that represents all the grades for “Malfoy, *D*.”

Each assignment is designated by a column,

2. Below, copy the column that represents all the grades on the “Love philter” assignment.

I know what you’re thinking, this is so easy it seems pointless. Well, it’s going to stay easy until tomorrow. So bear with me.

The dimensions *in that order*. So a “ 10×20 ” matrix means 10 rows and 20 columns.

3. What are the dimensions of Dr. Snape’s gradebook matrix?

For two matrices to be equal,

4. What must x and y be, in order to make the following matrix equal to Dr. Snape’s gradebook matrix?

| | | | |
|-----|---------|---------|-----|
| 100 | 105 | 99 | 100 |
| 80 | $x + y$ | 85 | 85 |
| 95 | 90 | 0 | 85 |
| 70 | 75 | $x - y$ | 75 |
| 85 | 90 | 95 | 90 |

Finally, it is possible to add *subtract* matrices. But you can only do this when the matrices have the same dimensions!!! If two matrices do not have exactly the same dimensions, you cannot add or subtract them. If they do have the same dimensions, you add and subtract them just by adding or subtracting each individual cell.

As an example: Dr. Snape has decided that his grades are too high, and he needs to curve them downward. So he plans to subtract the following grade-curving matrix

$$\begin{array}{cccc} 5 & 0 & 10 & 0 \\ 5 & 0 & 10 & 0 \\ 5 & 0 & 10 & 0 \\ 10 & 5 & 15 & 5 \\ 5 & 0 & 10 & 0 \end{array}$$

5. Below, write the new grade matrix.

6. In the grade-curving matrix, all rows except the fourth one are identical. What is the effect of the different fourth row on the final grades?

Name: _____

Introduction to Matrices—Homework

1. In the following matrix...

$$\begin{bmatrix} 1 & 3 & 7 & 4 & 9 & 3 \\ 6 & 3 & 7 & 0 & 8 & 1 \\ 8 & 5 & 0 & 7 & 3 & 2 \\ 8 & 9 & 5 & 4 & 3 & 0 \\ 6 & 7 & 4 & 2 & 9 & 1 \end{bmatrix}$$

- What are the dimensions? $_ \times _$
 - Copy the second column here:
 - Copy the third row here:
 - Write another matrix which is equal to this matrix.
2. Add the following two matrices.

$$\begin{bmatrix} 2 & 6 & 4 \\ 9 & n & 8 \end{bmatrix} + \begin{bmatrix} 5 & 7 & 1 \\ 9 & -n & 3n \end{bmatrix} =$$

3. Add the following two matrices.

$$\begin{bmatrix} 2 & 6 & 4 \\ 9 & n & 8 \end{bmatrix} + \begin{bmatrix} 5 & 7 \\ 9 & -n \end{bmatrix} =$$

4. Subtract the following two matrices.

$$\begin{bmatrix} 2 & 6 & 4 \\ 9 & n & 8 \end{bmatrix} - \begin{bmatrix} 5 & 7 & 1 \\ 9 & -n & 3n \end{bmatrix} =$$

5. Solve the following equation for x and y . (That is, find what x and y must be for this equation to be true.)

$$\begin{bmatrix} 2x \\ 5y \end{bmatrix} + \begin{bmatrix} x+y \\ -6x \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \end{bmatrix}$$

6. Solve the following equation for x and y . (That is, find what x and y must be for this equation to be true.)

$$\begin{bmatrix} x+y \\ 3x-2y \end{bmatrix} + \begin{bmatrix} 4x-y \\ x+5y \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 7 & 9 \end{bmatrix}$$

Name: _____

Multiplying Matrices I

Just to limber up your matrix muscles, let's try doing the following matrix addition.

$$1. \begin{bmatrix} 2 & 5 & x \\ 3 & 7 & 2y \end{bmatrix} + \begin{bmatrix} 2 & 5 & x \\ 3 & 7 & 2y \end{bmatrix} + \begin{bmatrix} 2 & 5 & x \\ 3 & 7 & 2y \end{bmatrix} =$$

2. How many times did you add that matrix to itself?

3. Rewrite problem 1 as a *multiplication problem*. (Remember what multiplication means—adding something to itself a bunch of times!)

This brings us to the world of multiplying a matrix by a number.

Let's do another example. I'm sure you remember Professor Snape's grade matrix.

TABLE 11.2:

| | Poison | Cure | Love philter | Invulnerability |
|-----------------|--------|------|--------------|-----------------|
| Granger, H | 100 | 105 | 99 | 100 |
| Longbottom, N | 80 | 90 | 85 | 85 |
| Malfoy, D | 95 | 90 | 0 | 85 |
| Potter, H | 70 | 75 | 70 | 75 |
| Weasley, R | 85 | 90 | 95 | 90 |

Now, we saw how Professor Snape could lower his grades (which he loves to do) by subtracting a curve matrix, *multiplying* the entire matrix by a number. In this case, he is going to multiply his grade matrix by $\frac{9}{10}$. If we designate his grade matrix as $[S]$ then the resulting matrix could be written as $\frac{9}{10}[S]$. (*Remember that the cells in a matrix are numbers! So $[S]$ is just the grades, not the names.)

4. Below, write the matrix $\frac{9}{10}[S]$.

Finally, it's time for Professor Snape to calculate final grades. He does this according to the following formula: "Poison" counts 30%, "Cure" counts 20%, "Love philter" counts 15%, and the big final project on "Invulnerability" counts 35%. For instance, to calculate the final grade for "Granger, H " he does the following calculation: $(30\%)(100) + (20\%)(105) + (15\%)(99) + (35\%)(100) = 100.85$.

To make the calculations easier to keep track of, the Professor represents the various weights in his grading matrix

$$\begin{bmatrix} .3 \\ .2 \\ .15 \\ .35 \end{bmatrix}$$

The above calculation can be written very concisely as multiplying a row matrix by a column matrix,

$$[100 \quad 105 \quad 99 \quad 100] \begin{bmatrix} .3 \\ .2 \\ .15 \\ .35 \end{bmatrix} = [100.85]$$

A “row matrix” means a matrix that is just one row. A “column matrix” means... well, you get the idea. When a row matrix and a column matrix have the same number of items, you can multiply the two matrices. What you do is, you multiply both of the first numbers, and you multiply both of the second numbers, and so on... and you add all those numbers to get one big number. The final answer is not just a number—it is a 1×1 matrix, with that one big number inside it.

5. Below, write the matrix multiplication that Professor Snape would do to find the grade for “Potter, *H*.” Show both the *problem* (the two matrices being multiplied) and the *answer* (the 1×1 matrix that contains the final grade).

Name: _____

Homework—Multiplying Matrices I

1. Multiply.

$$\frac{1}{2} \begin{bmatrix} 2 & 6 & 4 \\ 9 & n & 8 \end{bmatrix}$$

2. Multiply.

$$3 \begin{bmatrix} 2 & 3 & 4 \\ -6 & & 7 \end{bmatrix}$$

3. Multiply.

$$\begin{bmatrix} 3 & 6 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

4. Solve for x .

$$2 \begin{bmatrix} 7 & x & 3 \end{bmatrix} \begin{bmatrix} x \\ x \\ 5 \end{bmatrix} = [6]$$

Name: _____

Multiplying Matrices II

Just for a change, we’re going to start with... Professor Snape’s grade matrix!

TABLE 11.3:

| | Poison | Cure | Love philter | Invulnerability |
|----------------------|--------|------|--------------|-----------------|
| Granger, <i>H</i> | 100 | 105 | 99 | 100 |
| Longbottom, <i>N</i> | 80 | 90 | 85 | 85 |
| Malfoy, <i>D</i> | 95 | 90 | 0 | 85 |
| Potter, <i>H</i> | 70 | 75 | 70 | 75 |
| Weasley, <i>R</i> | 85 | 90 | 95 | 90 |

As you doubtless recall, the good Professor calculated final grades by the following computation: “Poison” counts 30%, “Cure” counts 20%, “Love philter” counts 15%, and the big final project on “Invulnerability” counts 35%. He was able to represent each student’s final grade as the product of a row matrix (for the student) times a column matrix (for weighting).

1. Just to make sure you remember, write the matrix multiplication that Dr. Snape would use to find the grade for “Malfoy, *D*.” Make sure to include both the two matrices being multiplied, and the final result!

I’m sure you can see the problem with this, which is that you have to write a separate matrix multiplication problem for every student. To get around that problem, we’re going to extend our definition of matrix multiplication so that the first matrix no longer has to be a row—it may be many rows. Each row of the first matrix becomes a new row in the answer. So, Professor Snape can now multiply his entire student matrix by his weighting matrix, and out will come a matrix with all his grades! Let’s try it. Do the following matrix multiplication. The answer will be a 3×1 matrix with the final grades for “Malfoy, *D*,” “Potter, *H*,” and “Weasley, *R*.”

$$2. \begin{bmatrix} 95 & 90 & 0 & 85 \\ 70 & 75 & 70 & 75 \\ 85 & 90 & 95 & 90 \end{bmatrix} \begin{bmatrix} .3 \\ .2 \\ .15 \\ .35 \end{bmatrix} =$$

OK, let’s step back and review where we are. Yesterday, we learned how to multiply a row matrix times a column matrix. Now we have learned that you can add more rows

For full generality of matrix multiplication, you just need to know this: if you add more columns 3×2 matrix where each row is a different *student* and each column is a different *weighting scheme*. Got all that? Give it a try now!

$$3. \begin{bmatrix} 95 & 90 & 0 & 85 \\ 70 & 75 & 70 & 75 \\ 85 & 90 & 95 & 90 \end{bmatrix} \begin{bmatrix} .3 & .4 \\ .2 & .2 \\ .15 & .3 \\ .35 & .1 \end{bmatrix} =$$

Name: _____

Homework—Multiplying Matrices II

1. Matrix $[A]$ is $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Matrix $[B]$ is the product $\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$.

- Find the product AB .
- Find the product BA .

2. Multiply.

$$\begin{bmatrix} 2 & 6 & 4 \\ 9 & 5 & 8 \end{bmatrix} \begin{bmatrix} 2 & 5 & 4 & 7 \\ 3 & 4 & 6 & 9 \\ 8 & 4 & 2 & 0 \end{bmatrix}$$

3. Multiply.

$$\begin{bmatrix} 2 & 5 & 4 & 7 \\ 3 & 4 & 6 & 9 \\ 8 & 4 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 & 6 & 4 \\ 9 & 5 & 8 \end{bmatrix}$$

4. $\begin{bmatrix} 5 & 3 & 9 \\ 7 & 5 & 3 \\ 2 & 7 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$

a. Multiply.

b. Now, multiply $\begin{bmatrix} 5 & 3 & 9 \\ 7 & 5 & 3 \\ 2 & 7 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 10 \\ 5 \end{bmatrix}$ —but not by manually multiplying it out! Instead, plug $x = 2$, $y = 10$, and $z = 5$ into the formula you came up with in part (a).

5. Multiply.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

6. $3 \begin{bmatrix} 3 & -2 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}$

a. Find the x and y values that will make this matrix equation true.

b. Test your answer by doing the multiplication to make sure it works out.

7. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} \text{Some} \\ \text{Matrix} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

a. Find the “some matrix” that will make this matrix equation true.

b. Test your answer by doing the multiplication to make sure it works out.

Name: _____

The “Identity” and “Inverse” Matrices

This assignment is brought to you by one of my favorite numbers, and I’m sure it’s one of yours... the number 1. *Some people say that 1 is the loneliest number that you’ll ever do. (Bonus: who said that?) But I say, 1 is the multiplicative identity.*

Allow me to demonstrate.

1. $5 \times 1 =$

2. $1 \times \frac{2}{3} =$

3. $-\pi \times 1 =$

4. $1 \times x =$

You get the idea? 1 is called the *multiplicative identity* because it has this lovely property that whenever you multiply it by anything, you get that same thing back. But that’s not all! Observe...

5. $2 \times \frac{1}{2} =$

6. $\frac{-2}{3} \times \frac{-3}{2} =$

The fun never ends! The point of all that was that every number has an 1.

7. Write the equation that defines two numbers a and b as inverses of each other.

8. Find the inverse of $\frac{4}{5}$.

9. Find the inverse of -3 .

10. Find the inverse of x .

11. Is there any number that does not have 7?

So, what does all that have to do with matrices? (I hear you crying.) Well, we've

12. $\begin{bmatrix} 3 & 8 \\ -4 & 12 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} =$

13. $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & 8 \\ -4 & 12 \end{bmatrix} =$

Pretty nifty, huh? When you multiply $[I]$) *the multiplicative identity*.

Remember that matrix multiplication does not, in general, commute: that is, for any two matrices $[A]$ and $[B]$, the product AB is not necessarily the same as the product BA . But in this case, it is: $[I]$ times another matrix gives you that other matrix back no matter which order you do the multiplication in. This is a key part of the definition of I , which is...

Definition: The matrix I is defined as the multiplicative identity if it satisfies the equation:

$$AI = IA = A$$

Which, of course, is just a fancy way of saying what I said before. If you multiply I by any matrix,

14. We have just seen that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ acts as the multiplicative identity for a 2×2 matrix.

a. What is the multiplicative identity for a 3×3 matrix?

b. Test this identity to make sure it works.

c. What is the multiplicative identity for a 5×5 matrix? (I won't make you test this one...)

d. What is the multiplicative identity for a 2×3 matrix?

e. Trick question! There isn't one. You could write a matrix that satisfies $AI = A$, but it would not *also* satisfy $IA = A$ —that is, it would not commute, which we said was a requirement. Don't take my word for it, try it! The point is that only square matrices (*same number of rows as columns) have an identity matrix.

So what about those inverses? Well, remember that two numbers a and b are inverses if $ab = 1$. As you might guess, we're going to define two matrices A and B as inverses if $AB = [I]$. Let's try a few.

15. Multiply: $\begin{bmatrix} 2 & 2\frac{1}{2} \\ -1 & -1\frac{1}{2} \end{bmatrix} \begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix}$

16. Multiply: $\begin{bmatrix} 3 & 5 \\ -2 & -4 \end{bmatrix} \begin{bmatrix} 2 & 2\frac{1}{2} \\ -1 & -1\frac{1}{2} \end{bmatrix}$

You see? These two matrices are $[I]$. We will designate the inverse of a matrix as A^{-1} , which looks like an exponent, but isn't really, it just means inverse matrix—just as we used f^{-1} to designate an inverse function. Which leads us to...

Definition

The matrix A^{-1} is defined as the multiplicative inverse of A if it satisfies the equation:
 $A^{-1}A = AA^{-1} = I$ (*where I is the identity matrix)

Of course, only a square matrix can have an inverse, since only a square matrix can have an inverse. *Now we know what an inverse matrix does, but how do you find one?*

17. Find the inverse of the matrix $\begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$

a. Since we don't know the inverse yet, we will designate it as a bunch of unknowns: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ will be our inverse matrix. Write down the equation that defines this unknown matrix as our inverse matrix.

b. Now, in your equation, you had a matrix multiplication. Go ahead and do

c. Now, remember that when we set two matrices equal 2×2 matrices equal, we actually end up with four different equations. Write these four equations.

d. Solve for $a, b, c,$ and d .

e. So, write the inverse matrix A^{-1} .

f. Test this inverse matrix to make sure it works!

Name: _____

Homework: The “Identity” and “Inverse” Matrices

a. Matrix A is $\begin{bmatrix} 4 & 10 \\ 2 & 6 \end{bmatrix}$.

- Write the identity matrix I for Matrix A .
- Show that it works.
- Find the inverse matrix A^{-1} .
- Show that it works.

b. Matrix B is $\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$

- Can you find a matrix that satisfies the equation $BI = B$?
- Is this an identity matrix for B ? If so, demonstrate. If not, why not?

c. Matrix C is $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$. Write the identity matrix for C .

d. Matrix D is $\begin{bmatrix} 1 & 2 \\ 3 & n \end{bmatrix}$.

- Find the inverse matrix D^{-1} .
- Test it.

Name: _____

The Inverse of the Generic 2×2 Matrix

Today you are going to find the inverse of the generic 2×2 matrix. Once you have done that, you will have a formula that can be used to quickly find the inverse of any 2×2 matrix.

The generic 2×2 matrix, of course, looks like this:

$$[A] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since its inverse is unknown, we will designate the inverse like this:

$$[A^{-1}] = \begin{bmatrix} w & x \\ y & z \end{bmatrix}$$

Our goal is to find a formula for w in terms of our original variables a, b, c , and d . That formula must not have any w, x, y , or z in it, since those are unknowns! Just the original four variables in our original matrix $[A]$. Then we will find similar formulae for x, y , and z and we will be done.

Our approach will be the same approach we have been using to find an inverse matrix. I will walk you through the steps—after each step, you may want to check to make sure you’ve gotten it right before proceeding to the next.

1. Write the matrix equation that defines A^{-1} as an inverse of A .
2. Now, do the multiplication, so you are setting two matrices equal to each other.
3. Now, we have two 2×2 matrices set equal to each other. That means every cell must be identical, so we get four different equations. Write down the four equations.
4. Solve. Remember that your goal is to find four equations—one for w , one for x , one for y , and one for z where each equation has only the four original constants a, b, c , and d !
5. Now that you have solved for all four variables, write the inverse matrix A^{-1} .

$$A^{-1} =$$

6. As the final step, to put this in the form that it is most commonly seen in, note that all four terms have an $ad - bc$ in the denominator. (*Do you have a $bc - ad$ instead? Multiply the top and bottom by -1 !) we can write our answer much more simply if we *pull out* the common factor of $\frac{1}{ad-bc}$. (This is similar to “pulling out” a common term from a polynomial. Remember how we *multiply a matrix by a constant*? This is the same thing in reverse.) So rewrite the answer with that term pulled out.

$$A^{-1} =$$

You’re done! You have found the generic formula for the inverse of any 2×2 matrix. Once you get the hang of it, you can use this formula to find the inverse of any 2×2 matrix very quickly. Let’s try a few!

7. The matrix $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

- a. Find the inverse—not the long way, but just by plugging into the formula you found above.
- b. Test the inverse to make sure it works.

8. The matrix $\begin{bmatrix} 3 & 2 \\ 9 & 5 \end{bmatrix}$

- a. Find the inverse—not the long way, but just by plugging into the formula you found above.
- b. Test the inverse to make sure it works.

9. Can you write a 2×2 matrix that *has no inverse*?

Name: _____

Using Matrices for Transformation

You are an animator for the famous company Copycat Studios. Your job is to take the diagram of the “fish” below (whose name is Harpoona) and animate a particular scene for your soon-to-be-released movie.

In this particular scene, the audience is looking down from above on Harpoona who begins the scene happily floating on the surface of the water. Here is a picture of Harpoona as she is happily floating on the surface.

Here is the matrix that represents her present idyllic condition.

$$[H] = \begin{bmatrix} 0 & 10 & 10 & 0 \\ 0 & 0 & 5 & 0 \end{bmatrix}$$

1. Explain, in words, how this matrix represents her position. That is, how can this matrix give instructions to a computer on exactly how to draw Harpoona?
2. The transformation $\frac{1}{2}[H]$ is applied to Harpoona.
 - a. Write the resulting matrix below.
 - b. In the space below, draw Harpoona after this transformation.
 - c. In the space below, answer this question in words: in general, $\frac{1}{2}[H]$ do to a picture?
3. Now, Harpoona is going to swim three units to the left. Write below a general transformation that can be applied to any 2×4 matrix to move a drawing three units to the left.
4. Harpoona—in her original configuration before she was transformed in either way—now undergoes the transformation $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} [H]$.
 - a. Write the new matrix that represents Harpoona below.
 - b. In the space below, draw Harpoona after this transformation.
 - c. In the space below, answer this question in words: in general $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} [H]$ do to a picture?
5. Now: in the movie’s key scene, the audience is looking down from above on Harpoona who begins the scene happily floating on the surface of the water. As the scene progresses, our heroine spins around and around in a whirlpool as she is slowly being sucked down to the bottom of the sea. “Being sucked down” is represented visually, of course, by shrinking.
 - a. Write a single transformation that will rotate Harpoona by 90° and shrink her.
 - b. Apply this transformation four times to Harpoona’s original state, and compute the resulting matrices that represent her next four states.
 - c. Now draw all four states—preferably in different colors or something.

Name: _____

Homework: Using Matrices for Transformation

1. Harpoona’s best friend is a fish named Sam, whose initial position is represented by the matrix:

$$[S_1] = \begin{bmatrix} 0 & 4 & 4 & 0 & 0 & 4 \\ 0 & 0 & 3 & 3 & 0 & 3 \end{bmatrix}$$

Draw Sam.

2. When the matrix $T = \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}$

is multiplied by any matrix, it effects a powerful transformation $S_2 = T S_1$. (You may use 1.7 as an approximation for $\sqrt{3}$.)

3. Draw Sam's resulting condition, S_2 .

4. The matrix $T^{[U+0080][U+0093]1}$ will, of course, do the opposite of T . Find $T^{[U+0080][U+0093]1}$. (You can use the formula for the inverse matrix that we derived in class, instead of starting from first principles. But make sure to first multiply the $\frac{1}{2}$ into T , so you know what the four elements are!)

5. Sam now undergoes this transformation, so his new state is given by $S_3 = T^{[U+0080][U+0093]1} S_2$. Find S_3 and graph his new position.

6. Finally, Sam goes through $T^{[U+0080][U+0093]1}$ again, so his final position is $S_4 = T^{[U+0080][U+0093]1} S_3$. Find and graph his final position.

7. Describe in words: what do the transformations T and $T^{[U+0080][U+0093]1}$ do, in general, to any shape?

Name: _____

Homework: Calculators

a. Solve on a calculator: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$

b. Solve on a calculator: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} - \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$

c. Solve on a calculator: $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$

d. Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 8 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$.

e. Multiply the matrix $\begin{bmatrix} 1 & 2 & 8 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ by its own inverse. What do you get? Is it what you expected?

f. Matrix A is $\begin{bmatrix} 6 & 1 & 2 \\ 0 & 9 & 5 \end{bmatrix}$. Matrix B is $\begin{bmatrix} 10 & -2 & 3 \\ 4 & 7 & 0 \end{bmatrix}$. Matrix C is $\begin{bmatrix} -6 & 12 \\ 9 & 7 \\ 3 & 2 \end{bmatrix}$. Use your calculator to find...

- AC
- CA
- $(A+B)C$
- $2A + \frac{1}{2}B$

Name: _____

Solving Linear Equations

I'm sure you remember our whole unit on solving linear equations... by graphing, by substitution, and by elimination. Well, now we're going to find a new way of solving those equations... by using matrices!

Oh, come on... why do we need another way when we've already got three?

Glad you asked! There are two reasons. First, this new method can be done entirely on a calculator. Cool! We like calculators.

Yeah, I know. But here's the even better reason. Suppose I gave you three equations with three unknowns, and asked you to do that. *Um... it would take a while.* How about four equations with four unknowns? *Please don't do that.* With matrices and your calculator, all of these are just as easy as two. *Wow! Do those really come up in real life?* Yes, all the time. Actually, this is just about the only "real-life" application I can give you for matrices, although there are also a lot of other ones. But solving many simultaneous equations is incredibly useful. *Do you have an example?* Oh, look at the time! I have to explain how to do this method.

So, here we go. Let's start with a problem from an earlier homework assignment. I gave you this matrix equation:

$$\begin{bmatrix} 9 & -6 \\ 18 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 9 \\ -3 \end{bmatrix}$$

1. The first thing you had to do was to rewrite this as two equations with two unknowns. Do that now. (Don't bother solving for x and y , just set up the equations.)

The point is that one matrix equation *two simultaneous equations*. What we're interested in doing is that process in *reverse*: I'll give you simultaneous equations, and you'll turn them into a matrix equation that represents the same thing. Let's try a few.

2. Write a single matrix equation that represents the two equations:

$$\begin{aligned} 3x + y &= -2 \\ 6x - 2y &= 12 \end{aligned}$$

3. Now, let's look at three

$$\begin{aligned} 7a + b + 2c &= -1 \\ 8a - 3b &= 12 \\ a - b + 6c &= 0 \end{aligned}$$

a. Write a single matrix equation that represents these three equations.

b. Just to make sure it worked, multiply it out and see what three equations you end up with.

OK, by now you are convinced that we can take simultaneous linear equations and rewrite them as a single matrix equation. In each case, the matrix equation looks like this:

$$AX = B$$

where A is a big square matrix, and X and B are column matrices. X is the matrix that we want to solve for—that is, it has all our variables in it, so if we find what X is, we find what our variables are. (For instance, in that last example, X was $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$.) So how do you solve something like this for X ? Time for some matrix algebra! We can't divide both sides by A , because we have not defined matrix division. But we can do the next best thing.

4. Take the equation $AX = B$, where A , X , and B are all matrices. Multiply both sides by A^{-1} (the inverse of A) in front. (Why did I say, "in front?" Remember that *order matters* when multiplying matrices. If we put A^{-1} in front of both sides, we have done the same thing to both sides.)

5. Now, we have $A^{-1}A$... gee, didn't that equal something? Oh, yeah... rewrite the equation simplifying that part.
6. Now, we're multiplying I by something... what does that do again? Oh, yeah... rewrite the equation again a bit simpler.

We're done! We have now solved for the matrix X .

So, what good is all that again?

Oh, yeah... let's go back to the beginning. Let's say I gave you these two equations:

$$\begin{aligned}3x + y &= -2 \\6x - 2y &= 12\end{aligned}$$

You showed in 2 how to rewrite this as one matrix equation $AX = B$. And you just found in 6 how to solve such an equation for X . So go ahead and plug A and B into your calculator, and then use the formula to ask your calculator directly for the answer!

7. Solve those two equations for x and y by using matrices on your calculator.

Did it work? x and y values into the original equations and make sure they work.

8. Check your answer to 7.

9. Now, solve the three simultaneous equations from 3 on your calculator, and check the answers.

Name: _____

Homework—Solving Linear Equations

- a. $4x + 2y = 3$
 $3y + 8x = 8$
- Solve these two equations by either substitution or elimination.
 - Now, rewrite those two equations as a matrix equation.
 - Solve the matrix equation. Your answer should be in the form of a matrix equation: $[X] =$
 - Now, using your calculator, find the numbers for your equation in part (c). Do they agree with the answers you found in part (a)?
- b. $6x - 8y = 2$
 $9x - 12y = 5$
- Solve by using matrices on your calculator.
 - Hey, what happened? Why did it happen, and what does it tell you about these two equations?
- c. $3x + 4y - 2z = 1$
 $8x + 3y + 3z = 4$
 $x - y + z = 7$
- Solve.
 - Check your answers.
- d. $2x - 5y + z = 1$
 $6x - y + 2z = 4$
 $4x - 10y + 2z = 2$
- Solve.
 - Check your answers.
- e. $3x + 3y - 2z = 4$
 $x - 7y + 3z = 9$
 $5x + 2z = 6$

- a. Solve. (Hey, there's no y is 0.)
 b. Check your answers.
- f. $3w + 4x - 8y + 2z = 4$
 $7w - 9x - 3y + 4z = 2$
 $2w + 5x + 2y - 10z = 7$
 $8w + 3x - 6y - z = 6$

- a. Solve.
 b. Check your answers.

Name: _____

Sample Test: Matrices

- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - 2 \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. What are $a, b, c,$ and d ?
- $\begin{bmatrix} 1 & 3 & 4 & n \end{bmatrix} \begin{bmatrix} 2 \\ 5 \\ 6 \\ 7 \end{bmatrix} =$
- $\begin{bmatrix} 4 & -2 \\ 0 & 3 \\ n & 1 \end{bmatrix} \begin{bmatrix} 3 & 6 & 8 \\ 2 & 9 & 0 \\ 1 & -1 & 4 \\ 0 & 4 & 2 \end{bmatrix} =$
- $\begin{bmatrix} 3 & 6 & 8 \\ 2 & 9 & 0 \\ 1 & -1 & 4 \\ 0 & 4 & 2 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 0 & 3 \\ n & 1 \end{bmatrix} =$
- $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} \text{some} \\ \text{matrix} \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$. What is "some matrix"?
- $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} \text{some} \\ \text{matrix} \end{bmatrix} = \begin{bmatrix} c & b & a \\ f & e & d \\ i & h & g \end{bmatrix}$. What is "some matrix"?
- Write two matrices that can be added *can be multiplied*.
 - Write two matrices that cannot be added or multiplied.
 - Write two matrices that can be added *cannot be multiplied*.
 - Write two matrices that can be multiplied *cannot be added*.
- Find the inverse of the matrix $\begin{bmatrix} 4 & x \\ 1 & -2 \end{bmatrix}$ by using the definition of an inverse matrix. *Note: if you are absolutely flat stuck on part (a), ask for the answer. You will receive no credit for part (a) but you may then be able to go on to parts (b) and (c).*
 - Test it, by showing that it fulfills the definition of an inverse matrix.
 - Find the inverse of the matrix $\begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$ by plugging $x = 3$ into your answer to part (a).
- Suppose $A, B, C, D,$ and E are matrices. Solve the following equation for C . $ABC = DE$
- Here are two equations and two unknowns.

$$\begin{aligned}6m + 2n &= -2 \\ -3m - n &= 1\end{aligned}$$

- Rewrite this problem as a matrix equation.
 - Solve. What are m and n ?
11. Solve the following equations for $a, b, c,$ and d .

$$\begin{aligned}2a + 3b - 5c + 7d &= 8 \\ 3a - 4b + 6c + 8d &= 10 \\ 10a + c + 6d &= 3 \\ a - b - c - d &= 69\end{aligned}$$

Extra credit: 2×2 matrix that has no inverse. No two of the four numbers should be the same.

CHAPTER **12**

Modeling Data with Functions

Chapter Outline

12.1 MODELING DATA WITH FUNCTIONS

12.1 Modeling Data with Functions

Name: _____

Direct Variation

1. Suppose I make \$6/hour. Let t represent the number of hours I work, and m represent the money I make.

a. Make a table showing different t values and their corresponding m values. (m is not how much money I make in that particular hour—it's how much total money I have made, after working that many hours.)

TABLE 12.1:

| time (t) | money (m) |
|--------------|---------------|
|--------------|---------------|

b. Which is the dependent variable, and which is the independent variable?

c. Write the function.

d. Sketch a quick graph of what the function looks like.

e. In general: if I double the number of hours, what happens to the amount of money?

2. I am stacking bricks to make a wall. Each brick is 4" high. Let b represent the number of bricks, and h represent the height of the wall.

a. Make a table showing different b values and their corresponding h values.

TABLE 12.2:

| bricks (b) | height (h) |
|----------------|----------------|
|----------------|----------------|

b. Which is the dependent variable, and which is the independent variable?

c. Write the function.

d. Sketch a quick graph of what the function looks like.

e. In general: if I triple the number of bricks, what happens to the height?

3. The above two scenarios are examples of direct variation. y “varies directly” with x , then it can be written as a function $y = kx$, where k is called the *constant of variation*. (We also sometimes say that “ y is proportional to x ,” where k is called the constant of proportionality. Why do we say it two different ways? Because, as you’ve always suspected, we enjoy making your life difficult. Not “students in general,” but just you personally.) So, *if y varies directly with x ...*

a. What happens to y if x doubles? (*Hint*: You can find and prove the answer from the equation $y = kx$.)

b. What happens to y if x is cut in half?

c. What does the graph $y(x)$ look like? What does k represent in this graph?

4. Make up a word problem like numbers (1) and (2) above, on the subject of *fast food*. Your problem should *not* involve getting paid or stacking bricks. It *should* involve two variables that vary directly with each other. Make up the scenario, define the variables, and then do problems a-e exactly like my two problems.

Name: _____

Homework Inverse Variation

1. An astronaut in space is performing an experiment with three balloons. The balloons are all different sizes, *same amount of air* in them. As you might expect, the balloons that are very small experience a great deal of air pressure (the air inside pushing out on the balloon); the balloons that are very large, experience very little air pressure. He measures the volumes and pressures and comes up with the following chart.

TABLE 12.3:

| Volume (V) | Pressure (P) |
|----------------|------------------|
| 5 | 270 |
| 10 | 135 |
| 15 | 90 |
| 20 | $67\frac{1}{2}$ |

- Which is the dependent variable, and which is the independent variable?
 - When the volume doubles
 - When the volume triples,
 - Based on your answers to parts (a) - (c), what would you expect?
 - On the right of the table add a third column that represents the quantity PV : pressure times volume. Fill in all four values for this quantity. What do you notice about them?
 - Plot all four points on the graph paper, and fill in a sketch of what the graph looks like.
 - Write the function $P(V)$. Make sure that it accurately gets you from the first column to the second in all four instances! (Part (e) is a clue to this.)
 - Graph your function $P(V)$ on your calculator, and copy the graph onto the graph paper. Does it match your graph in part (f)?
2. The three little pigs have built three houses—made from straw, Lincoln Logs[®], and bricks, respectively. Each house is 20' high. The pieces of straw are $\frac{1}{10}$ thick; the Lincoln Logs[®] are 1 thick; the bricks are 4 thick. Let t be the thickness of the building blocks, and let n be the number of such blocks required to build a house 20' high. (*Note*: There are 12 in 1'. But you probably knew that . . .)

- Make a table showing different t values and their corresponding n values.

TABLE 12.4:

| Building Blocks | thickness (t) | number (n) |
|---------------------------|-------------------|----------------|
| Straw | | |
| Lincoln Logs [®] | | |
| Bricks | | |

- Which is the dependent variable, and which is the independent variable?
- When the thickness of the building blocks doubles,
(*Not sure? Pretend that the pig's cousin used 8 logs, and his uncle used 16 logs. See what happens to the number required as you go up in this sequence. . .)
- When the thickness of the building blocks is halved,
- On the right of the table add a fourth column, that represents the quantity tn : thickness times number. Fill in all three values for this quantity. What do you notice about them? What do they actually represent, in our problem?
- Plot all three points on the graph paper, and fill in a sketch of what the graph looks like.

- g. Write the function $n(t)$.
- h. Graph your function $n(t)$ on your calculator, and copy the graph onto the graph paper. Does it match your graph in part (f)?
3. The above two scenarios are examples of inverse variation. y , “varies inversely” with x , then it can be written as a function $y = \frac{k}{x}$, where k is called the *constant of variation*. So, *if y varies inversely with x* . . .
- a. What happens to y if x doubles? (*Hint*: You can find and prove the answer from the equation $y = \frac{k}{x}$)
- b. What happens to y if x is cut in half?
- c. What does the graph $y(x)$ look like? What happens to this graph when k increases? (* You may want to try a few different ones on your calculator to see the effect k has.)
4. Make up a word problem like numbers (1) and (2) above. Your problem should *not* involve pressure and volume, or building a house. It *should* involve two variables that vary inversely with each other. Make up the scenario, define the variables, and then do problems a-h exactly like my two with each other. Make up the scenario, define the variables, and then do problems a-h exactly like my two problems.

Name: _____

Homework Direct and Inverse Variation

For questions

1. For the following set of data . . .

TABLE 12.5:

| x | y |
|-----|-----|
| 3 | 5 |
| 6 | 11 |
| 21 | 34 |

- a. Does it represent direct variation, inverse variation, or neither?
- b. If it is direct or inverse, what is the constant of variation?
- c. If $x = 30$, what would y be?
- d. Sketch a quick graph of this relationship.
2. For the following set of data . . .

TABLE 12.6:

| x | y |
|-----|-----|
| 3 | 18 |
| 4 | 32 |
| 10 | 200 |

- a. Does it represent direct variation, inverse variation, or neither?
- b. If it is direct or inverse, what is the constant of variation
- c. If $x = 30$, what would y be?
- d. Sketch a quick graph of this relationship.
3. For the following set of data . . .

TABLE 12.7:

| x | y |
|-----|-----|
| 3 | 20 |
| 6 | 10 |
| 21 | 3 |

- Does it represent direct variation, inverse variation, or neither?
 - If it is direct or inverse, what is the constant of variation?
 - If $x = 30$, what would y be?
 - Sketch a quick graph of this relationship.
4. In number 2 above, as you (hopefully) saw, the relationship is neither direct nor inverse. However, the relationship can be expressed this way: y is directly proportional to x^2 . Write the function that indicates this relationship. What is k ? one table is miss
5. In June, 2007, Poland argued for a change to the voting system in the European Union Council of Ministers. The Polish suggestion: each member's voting strength should be directly proportional to the square root of his country's population.

I swear I am not making this up.

Also in the category of "things I am not making up," the following table of European Populations comes from Wikipedia.

| | |
|-----------|------------|
| Germany | 83,251,851 |
| Italy | 59,715,625 |
| Poland | 38,625,478 |
| Luxemberg | 48,569 |

- Write an equation that represents Pensore's Rule. Be sure to clearly label your variables.
- Suppose that Pensore's Rule was followed, and suppose that Poland voting strength was exactly 100 (which I did actually make up, but of course it doesn't matter). What would the voting strength of Germany, Italy, and Luxembourg be?

6. Write a "real world" word problem involving an inverse relationship, on the topic of movies

7. Joint Variation: The term "Joint Variation" is used to indicate that *one variable varies directly as two different variables*. This is illustrated in the following example.

Al is working as a waiter. When a group of people sit down at a table, he calculates his expected tip (T) as follows: multiply the *number of people* (N), times the *average meal cost* (C), times 0.15 (for a 15% tip).

- If the number of people at the table doubles, does Al's expected tip double?
- If the average cost per meal doubles, does Al's expected tip double?
- Write the function that expresses the dependent variable, T , as a function of the *two* independent variables, N and C .
- Write the general function that states "z varies jointly as both x and y ." Your function will have an unknown k in it, a constant of variation.

8. **Light Intensity:** Stacy visits a tanning booth, where she spends several hours with lamps shining on her skin, thus giving her a beautiful copper-colored tan and a sharply increased risk of skin cancer. For reasons known only to

herself, she considers this a good trade-off. Anyway, Stacy has a lot of time to just lie there and think, and she starts to consider the question: which bulb is shining on her skin with the most intensity? The answer is that the intensity I of a bulb *varies directly* with the strength S of the bulb, and *varies inversely* with the square of the distance d of the bulb from her skin.

- Bulbs A and B are the same distance away, but bulb B is twice as strong as bulb A . If bulb A shines with an intensity of 5, what is the intensity of bulb B ?
- Bulbs A and C are the same strength as each other, but bulb A is twice as far away from Stacy as bulb C . If bulb A shines with an intensity of 5, what is the intensity of bulb B ?
- Write a function to represent the statement “the intensity I of a bulb varies directly with the strength S of the bulb, and varies inversely with the square of the distance d of the bulb from Stacy’s skin.” Your function will have an unknown k in it, a constant of variation.

Name: _____

Homework: Calculator Regression

1. **Canadian Voters:** The following table shows the percentage of Canadian voters who voted in the 1996 federal election.

| | | | | | |
|--------|----|----|----|----|----|
| Age | 20 | 30 | 40 | 50 | 60 |
| %voted | 59 | 86 | 87 | 91 | 94 |

- Enter these points on your calculator lists.
 - Set the Window on your calculator so that the x -values go from 0 to 60, and the y -values go from 0 to 100. Then view a graph of the points on your calculator. Do they increase steadily (like a line), or increase slower and slower (like a log), or increase more and more quickly (like a parabola or an exponent)?
 - Use the [STAT] function on your calculator to find an appropriate function to model this data. Write that function below.
 - Graph the function on your calculator. Does it match the points well? Are any of the points “outliers?”
2. **Height and Weight:** A group of students record their height (in inches) and weight (in pounds). The results are on the table below.

| | | | | | | | | | | | | | |
|--------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| Height | 68 | 74 | 66 | 68 | 72 | 69 | 65 | 71 | 69 | 72 | 71 | 64 | 65 |
| weight | 180 | 185 | 150 | 150 | 200 | 160 | 125 | 220 | 220 | 180 | 190 | 120 | 110 |

- Enter these points on your calculator lists.
 - Set the Window on your calculator appropriately, and then view a graph of the points on your calculator. Do they increase steadily (like a line), or increase slower and slower (like a log), or increase more and more quickly (like a parabola or an exponent)?
 - Use the [STAT] function on your calculator to find an appropriate function to model this data. Write that function below.
 - Graph the function on your calculator. Does it match the points well? Are any of the points “outliers?”
3. **Gas Mileage:** The table below shows the weight (in hundreds of pounds) and gas mileage (in miles per gallon) for a sample of domestic new cars.

| | | | | | | | | | | |
|---------|----|----|----|----|----|----|----|----|----|----|
| Weight | 29 | 35 | 28 | 44 | 25 | 34 | 30 | 33 | 28 | 24 |
| mileage | 31 | 27 | 29 | 25 | 31 | 29 | 28 | 28 | 28 | 33 |

- Enter these points on your calculator lists.
- Set the Window on your calculator appropriately, and then view a graph of the points on your calculator. Do they decrease steadily (like a line), or decrease slower and slower (like a log), or decrease more and more quickly (like a parabola or an exponent)?
- Use the [STAT] function on your calculator to find an appropriate function to model this data. Write that function below.

d. Graph the function on your calculator. Does it match the points well? Are any of the points “outliers?”

4. **TV and GPA:** A graduate student named Angela Hershberger at Indiana University-South Bend did a study to find the relationship between TV watching and Grade Point Average among high school students. Angela interviewed 50 high school students, turning each one into a data point, where the independent (x) value was the number of hours of television watched per week, and the dependent (y) value was the high school grade point average. (She also checked the *types* of television watched—*eg* news or sitcoms—and found that it made very little difference. Quantity, not quality, mattered.)

In a study that you can read all about at www.iusb.edu/journal/2002/hershberger/hershberger.html, Angela found that her data could best be modeled by the linear function $y = [U+0080] [U+0093]0.0288x + 3.4397$. Assuming that this line is a good fit for the data

- What does the number 3.4397 tell you? (Don’t tell me about lines and points: tell me about students, TV, and grades.)
- What does the number $[U+0080] [U+0093]0.0288$ tell you? (Same note.)

Name: _____

Sample Test: Modeling Data with Functions

1. Three cars and an airplane are traveling to New York City. But they all go at different speeds, so they all take different amounts of time to make the 500–mile trip. Fill in the following chart.

TABLE 12.8:

| Speed(s) - miles per hour | Time(t) - hours |
|---------------------------|-----------------|
| 50 | |
| 75 | |
| 100 | |
| 500 | |

- Is this an example of direct variation, inverse variation, or neither of the above?
 - Write the function $s(t)$.
 - If this is one of our two types, what is the constant of variation?
2. There are a bunch of squares on the board, of different sizes.

TABLE 12.9:

| s - length of teh side of a square | A - area of the square |
|------------------------------------|------------------------|
| 1 | |
| 2 | |

TABLE 12.9: (continued)

| s - length of the side of a square | A - area of the square |
|---|-------------------------------|
| 3 | |
| 4 | |

- Is this an example of direct variation, inverse variation, or neither of the above?
 - Write the function $A(s)$.
 - If this is one of our two types, what is the constant of variation?
3. Anna is planning a party. Of course, as at any good party, there will be a lot of on hand! 50 Coke cans fit into one recycling bin. So, based on the amount of Coke she buys, Anna needs to make sure there are enough recycling bins.

TABLE 12.10:

| c-Coke cans Anna buys | b - recycling bins she will need |
|------------------------------|---|
| 50 | |
| 100 | |
| 200 | |
| 400 | |

- Is this an example of direct variation, inverse variation, or neither of the above?
 - Write the function $b(c)$.
 - If this is one of our two types, what is the constant of variation?
4. Make up a word problem involving inverse variation, on the topic of skateboarding.
- Write the scenario.
 - Label and identify the independent and dependent variables.
 - Show the function that relates the dependent to the independent variable. This function should (of course) be an inverse relationship, and it should be obvious from your scenario!
5. I found a Web site (this is true, really) that contains the following sentence:
 [This process] introduces an additional truncation error [directly] proportional to the original error and inversely proportional to the gain (g) and the truncation parameter (q).
 I don't know what most of that stuff means any more than you do. But if we use T for the "additional truncation error" and E for the "original error," write an equation that expresses this relationship.
6. Which of the following correctly expresses, in words, the relationship of the area of a circle to the radius?
- The area is directly proportional to the radius
 - The area is directly proportional to the square of the radius
 - The area is inversely proportional to the radius
 - The area is inversely proportional to the square of the radius
7. Now, suppose we were to write the inverse
 The radius of a circle is _____ proportional to _____ the area.
8. **Death by Cholera:** In 1852, William Farr reported a strong association between low elevation and deaths from cholera. Some of his data are reported below.

| | | | | | | | |
|-------------------------------|-----|----|----|----|----|-----|-----|
| E:Elevation(ft) | 10 | 30 | 50 | 70 | 90 | 100 | 350 |
| C:Cholera morality(per10,000) | 102 | 65 | 34 | 27 | 22 | 17 | 8 |

a. Use your calculator to create the following models, and write the appropriate functions $C(E)$ in the blanks.

Linear: $C =$

Quadratic: $C =$

Lograthamic: $C =$

exponential: $C =$

b. Which model do you think is the best? Why?

c. Based on his very strong correlation, Farr concluded that bad air had settled into low-lying areas, causing outbreaks of cholera. We now know that air quality has nothing to do with causing cholera: the water-borne bacterial *Vibrio cholera*

CHAPTER

13

Analytic Geometry (or...) Conic Sections

Chapter Outline

13.1 ANALYTIC GEOMETRY (OR...) CONIC SECTIONS

13.1 Analytic Geometry (or...) Conic Sections

Name: _____

Distance

- Draw the points $(2, 5)$, $(10, 5)$, and $(10, 1)$, and the triangle they all form.
- Find the distance from $(2, 5)$ to $(10, 5)$ (just by looking at it).
- Find the distance from $(10, 5)$ to $(10, 1)$ (just by looking at it).
- Find the distance from $(2, 5)$ to $(10, 1)$, using your answers to (2) and (3) and the Pythagorean Theorem.
- I start at Raleigh Charter High School. I drive 5 miles West, and then 12 miles North. How far am I now from RCHS? (*Hint*: the answer is not 17 miles. Draw it!)
- Draw a point anywhere in the first quadrant. Instead of labeling the specific coordinates of that point, just label it (x, y) .
- How far down x -axis? (*Hint*: Think about specific points—such as $(4, 2)$ or $(1, 10)$ —until you can see the pattern and answer the question for the general point (x, y) .)
- How far across y -axis?
- Find the distance d from the origin to that point (x, y) , using the Pythagorean Theorem. This will give you a formula for the distance from *any* point, to the origin.
- Find the distance from the point $(3, 7)$ to the line $y = 2$.
- Find the distance from the generic point (x, y) (as before) to the line $y = 2$.
- Find the distance from the point $(3, 7)$ to the line $y = -2$.
- Find the distance from the generic point (x, y) to the line $y = -2$.
- I'm thinking of a point which is exactly 5 units away from the point $(0, 0)$. The y -coordinate of my point is 0. What is the x -coordinate? Draw this point.
- I'm thinking of two points which are exactly 5 units away from $(0, 0)$. The x -coordinates of both points is 4. What are the y -coordinates? Draw these points on the same graph that you did in 14.
- I'm thinking of two points which are exactly 5 units away from $(0, 0)$. The x -coordinates of both points is -4 . What are the y -coordinates? Draw these points on the same graph that you did in 14.

Name: _____

Homework: Distance

- Draw a point anywhere. Instead of labeling the specific coordinates of that point, just label it (x_1, y_1) .
- Draw another point somewhere else. Label it (x_2, y_2) . To make life simple, make this point higher and to the right of the first point.
- Draw the line going from (x_1, y_1) to (x_2, y_2) . Then fill in the other two sides of the triangle
- How far up
- How far across
- Find the distance d from (x_1, y_1) to (x_2, y_2) , using the Pythagorean Theorem. This will give you a general formula for the distance between any two points.
- Plug in $x_2 = 0$ and $y_2 = 0$ into your formula. You should get the same formula you got on the previous assignment, for the distance between any point and the origin. Do you?
- Draw a line from $(0, 0)$ to $(4, 10)$. Draw the point at the *exact middle* of that line. (Use a ruler if you have to.) What are the coordinates of that point?

- i. Draw a line from $(-3, 2)$ to $(5, -4)$. What are the coordinates of the midpoint?
- j. Look back at your diagram of a line going from (x_1, y_1) to (x_2, y_2) . What are the coordinates of the midpoint of that line?
- k. Find the distance from the point $(3, 7)$ to the line $x = 2$.
 - l. Find the distance from the generic point (x, y) to the line $x = 2$.
- m. Find the distance from the point $(3, 7)$ to the line $x = -2$.
- n. Find the distance from the generic point (x, y) to the line $x = -2$.
- o. Find the coordinates of all the points that have y -coordinate 5, and which are exactly 10 units away from the origin.
- p. Draw all the points you can find which are exactly 3 units away from the point $(4, 5)$.

Name: _____

All The Points Equidistant from a Given Point

1. Draw as many points as you can which are exactly 5 units away from $(0, 0)$, and fill in the shape. What shape is it?
2. Now, let's see if we can find the equation (x, y) to be on the shape, it must be exactly five units away from the origin. So we have to take the sentence:

The point

and translate it into math. Then we will have an equation that describes every point on our shape,

OK, but how do we do that?

- a. Above is a drawing of our point (x, y) , 5 units away from the origin. On the drawing, I have made a little triangle as usual. How long is the vertical line on the right side of the triangle? Label it in the picture.
- b. How long is the horizontal line at the bottom of the triangle? Label it in the picture.
- c. Now, all three sides are labeled. Just write down the Pythagorean Theorem for this triangle, and you have the equation for our shape!
- d. Now, let's see if it worked. A few points that are obviously 5 units away from the origin—are the points $(5, 0)$ and $(4, -3)$. Plug them both into your equation from the last part and see if they work.
- e. A few points that are clearly not $(1, 4)$ and $(-2, 7)$. Plug them both into your equation for the shape to make sure they don't work!

3. OK, that was all the points that were 5 units away from the origin. Now we're going to find an equation for the shape that represents all points that are exactly 3 units away from the point $(4, -1)$. Go through all the same steps we went through above—draw the point $(4, -1)$ and an arbitrary point (x, y) , draw a little triangle between them, label the distance from (x, y) to $(4, -1)$ as being 3, and write out the Pythagorean Theorem. Don't forget to test a few points!

4. By now you probably get the idea. So—without 7 units away from the point $(-5, 3)$.

5. And finally, the generalization as always: write down the equation for all the points that are exactly r units away from the point (h, k) .

Name: _____

Circles

Homework: Circles

(*aka "All the Points Equidistant from a Given Point")

- Write down the equation for a circle with center $(-3, -6)$ and radius 9. (Although the wording is different, this is exactly like the problems you did on the in-class assignment.)
- Now, let's take it the other way. $(x-4)^2 + (y+8)^2 = 49$ is the equation for a circle.
 - What is the center of the circle?
 - What is the radius?
 - Draw the circle.
 - Find two points on the circle (by looking at your drawing) and plug them into the equation to make sure they work. (Show your work!)

$2x^2 + 2y^2 + 8x + 24y + 100 = 0$ is also the equation for a circle. But in order to graph it, we need to put it into our canonical form $(x-h)^2 + (y-k)^2 = r^2$. In order to do that, we have to complete the square... twice! Here's how it looks.

| | |
|--|--|
| $2x^2 + 2y^2 + 8x + 24y + 100 = 0$ | The original problem |
| $x^2 + y^2 + 4x + 12y + 50 = 0$ | Divide by the coefficient of x^2 and y^2 |
| $(x^2 + 4x) + (y^2 + 12y) = -50$ | Collect x and y terms together, and bring the number to the other side |
| $(x^2 + 4x + 4) + (y^2 + 12y + 36) = -50 + 4 + 36$ | Complete the square in <u>both</u> parentheses |
| $(x+2)^2 + (y+6)^2 = 10$ | Done! The center is $(-2, -6)$ and the radius is $\sqrt{10}$ |

Got it? Now you try!

3. $3x^2 + 3y^2 + 18x + 30y + 6 = 0$

- Complete the square—as I did above—to put this into the form:

$$(x-h)^2 + (y-k)^2 = r^2.$$

- What are the center and radius of the circle?
 - Draw the circle.
 - Find two points on the circle (by looking at your drawing) and plug them into the original
- Name: _____

All the Points Equidistant from a Point and a Line

On the drawing below is the point $(0, 3)$ and the line $(y = -3)$. What I want you to do is to find all the points that are the *same distance* from $(0, 3)$ that they are from the line $(y = -3)$.

One of the points is very obvious. You can get two more of them, exactly, with a bit of thought.

After that you have to start playing around. Feel free to use some sort of measuring device (such as your fingernail, or a pencil eraser). When you think you have the whole shape, call me and let me look.

Name: _____

Parabolas

Homework: Vertical and Horizontal Parabolas

1. $y = 3x^2 - 30x - 70$

- Put into the standard form of a parabola.

- b. Vertex:
 - c. Opens (up / down / right / left):
 - d. Graph it
2. $x = y^2 + y$
- a. Put into the standard form of a parabola.
 - b. Vertex:
 - c. Opens (up / down / right / left):
 - d. Graph it
3. Find the equation for a parabola that goes through the points $(0, 2)$ and $(0, 8)$.

Name: _____

Parabolas: From Definition to Equation

We have talked about the geometric definition of a parabola: “all the points in a plane that are the same distance from a given point (the focus) *directrix*.” And we have talked about the general equations for a parabola:

Vertical parabola: $y = a(x - h)^2 + k$

Horizontal parabola: $x = a(y - k)^2 + h$

What we haven't done is connect these two things—the definition *equation* for a parabola. We're going to do it the exact same way we did it for a circle—start with the geometric definition and turn it into an equation.

In the drawing above, I show a parabola whose focus is the origin $(0, 0)$ and directrix is the line $y = -4$. On the parabola is a point (x, y) which represents *any point on the parabola*.

- a. d_1 is the distance from the point (x, y) to the focus $(0, 0)$. What is d_1 ?
- b. d_2 is the distance from the point (x, y) to the directrix $(y = -4)$. What is d_2 ?
- c. What defines the parabola as such—what makes (x, y) part of the parabola—is that these two distances are the *same*. Write the equation $d_1 = d_2$ and you have the parabola.
- d. Simplify your answer to 3; that is, rewrite the equation in the standard form.
- e. What does your equation say the vertex should be? Does it match the drawing?

Name: _____

Sample Test: Distance, Circles and Parabolas

1. Below are the points $(-2, 4)$ and $(-5, -3)$.
 - a. How far is it across
 - b. How far is it down
 - c. How far are the two points from each other?
 - d. What is the midpoint
2. What is the distance from the point $(-1024, 3)$ to the line $y = -1$?
3. Find all the points that are exactly 4 units away from the origin, where the x and y coordinates are the *same*. (If you get stuck, ask for a hint—it will cost you points, but it's better than nothing...)
4. $2x^2 + 2y^2 - 6x + 4y + 2 = 0$
 - a. Put this equation in the standard form for a circle.
 - b. What is the center?
 - c. What is the radius?

- d. Graph it on the graph paper.
- e. Find one point on your graph, and test it in the original equation. (No credit unless I can see your work!)
5. $x = -\frac{1}{4}y^2 + y + 2$
- a. Put this equation in the standard form for a parabola.
- b. What direction does it open in?
- c. What is the vertex?
- d. Graph it on the graph paper.
6. Find the equation for a circle where the center is the point $(-2, 5)$ and the radius is 3.
7. We're going to find the equation of a parabola whose focus is $(3, 2)$ and whose directrix is the line $x = -3$. But we're going to do it straight from the *definition* of a parabola.

In the drawing above, I show the focus and the directrix, and an arbitrary point (x, y) on the parabola.

- a. d_1 is the distance from the point (x, y) to the focus $(3, 2)$. What is d_1 ?
- b. d_2 is the distance from the point (x, y) to the directrix $(x = -3)$. What is d_2 ?
- c. What defines the parabola as such—what makes (x, y) part of the parabola—is that $d_1 = d_2$. Write the equation for the parabola.
- d. Simplify your answer to part (c); that is, rewrite the equation in the standard form.

Name: _____

Distance to This Point Plus Distance to That Point is Constant

On the drawing below are the points $(3, 0)$ and $(-3, 0)$. We're going to draw yet another shape—not a circle or a parabola or a line, which are the three shapes we know about. In order to be on our shape, the point (x, y) must have the following property:

The distance from (x, y) to $(3, 0)$, *plus* the distance from (x, y) to $(-3, 0)$, must equal 10.

We're going to start this one the same way we did our other shapes: intuitively. Your object is to find all the points that have that particular property. Four of them are... well, maybe none of them are exactly obvious, but there are four that you can get exactly, with a little thought. After that, you have to sort of figure it out as we did before.

When you think you know the shape, don't call it out! Call me over and I will tell you if it's right.

Name: _____

Ellipses

Homework: Ellipses

1. In class, we discussed how to draw an ellipse using a piece of cardboard, two thumbtacks, a string, and a pen or marker. Do this. (*Yes, this is a real part of your homework!*)
2. $\frac{(x-2)^2}{9} + \frac{y^2}{4} = 1$
- a. Is it horizontal or vertical?
- b. What is the center?
- c. What is a ?
- d. What is b ?

e. What is c ?

f. Graph it.

3. $\frac{4x^2}{9} + 25y^2 = 1$

This sort of looks like an ellipse in standard form, doesn't it? It even has a

a. Rewrite the left-hand term, $\frac{4x^2}{9}$, by dividing the top and bottom of the fraction by 4. Leave the bottom as a fraction, don't make it a decimal.

b. Rewrite the right-hand term, $25y^2$, by dividing the top and bottom of the fraction by 25. Leave the bottom as a fraction don't make it a decimal.

c. Now,

d. How long is the major axis?

e. How long is the minor axis?

f. What are the coordinates of the two foci?

g. Graph it.

4. $18x^2 + \frac{1}{2}y^2 + 108x + 5y + 170 = 0$

a. Put in standard form.

b. Is it horizontal or vertical?

c. What is the center?

d. How long is the major axis?

e. How long is the minor axis?

f. What are the coordinates of the two foci?

g. Graph it.

5. The major axis of an ellipse runs from $(5, -6)$ to $(5, 12)$. One focus is at $(5, -2)$. Find the equation for the ellipse.

6. The foci of an ellipse are at $(-2, 3)$ and $(2, 3)$ and the ellipse contains the origin. Find the equation for the ellipse.

7. We traditionally say that the Earth is 93 million miles away from the sun. However, if it were *always* 93 million miles away, that would be a circle (right?). In reality, the Earth travels in an *ellipse*, with the *sun at one focus*. According to one Web site I found,

There is a 6% difference in distance between the time when we're closest to the sun (perihelion) and the time when we're farthest from the sun (aphelion). Perihelion occurs on January 3 and at that point, the earth is 91.4 million miles away from the sun. At aphelion, July 4, the earth is 94.5 million miles from the sun.

(*Source: <http://geography.about.com/library/weekly/aa121498.htm>)

Write an equation to describe the orbit of the Earth around the sun. Assume that it is centered on the origin and that the major axis is horizontal. (*Why not? There are no axes in space, so you can put them wherever it is most convenient.) Also, work in units of *millions of miles*—so the numbers you are given are simply 91.4 and 94.5.

The Ellipse: From Definition to Equation

Here is the geometric definition of an ellipse. There are two points called the "foci": in this case, $(-3, 0)$ and $(3, 0)$. A point is on the ellipse if the *sum* of its distances to both foci is a certain constant: in this case, I'll use 10. Note that the foci *define* the ellipse, but are not *part of* it.

The point (x, y) represents any point on the ellipse. d_1 is its distance from the first focus, and d_2 to the second.

1. Calculate the distance d_1 (by drawing a right triangle, as always).

2. Calculate the distance d_2 (by drawing a right triangle, as always).
3. Now, to create the equation for the ellipse, write an equation asserting that the sum d_1 and d_2 equals 10.
Now simplify it. We did problems like this earlier in the year (radical equations, the “harder” variety that have two radicals). The way you do it is by isolating the square root, and then squaring both sides. In this case, there are two square roots, so you will need to go through that process twice.
4. Rewrite your equation in 3, isolating one of the square roots.
5. Square both sides.
6. Multiply out, cancel, combine, simplify. This is the big step! In the end, isolate the only remaining square root.
7. Square both sides again.
8. Multiply out, cancel, combine, and get it to look like the standard form for an ellipse.
9. Now, according to the “machinery” of ellipses, what should that equation look like? Horizontal or vertical? Where should the center be? What are a , b , and c ? Does all that match the picture we started with?

Name: _____

Distance to This Point Minus Distance to That Point is Constant

On the drawing below are the points $(5,0)$ and $(-5,0)$. We’re going to draw yet another shape—our final conic section. In order to be on our shape, the point (x,y) must have the following property:

Take the distance from (x,y) to $(5,0)$, and the distance from (x,y) to $(-5,0)$. Those two distances must *differ* by 6. (In other words, this distance *minus* that distance must equal ± 6 .)

We’re going to start this one the same way we did our other shapes: intuitively. Your object is to find all the points that have that particular property. Start by finding the two points on the x -axis that work. After that, you have to sort of work it out as we did before.

When you think you know the shape, don’t call it out! Call me over and I will tell you if it’s right.

Name: _____

Hyperbolas

Homework: Hyperbolas

1. Complete the following chart, showing the similarities and differences between ellipses and hyperbolas.

TABLE 13.1:

| | Ellipse | Hyperbola |
|--|--|-----------|
| How to identify an equation with this shape | Has an x^2 and a y^2 with <i>different coefficients</i> , but the <i>same sign</i> . $3x^2 + 2y^2$ for instance. | |
| Equation in standard form: horizontal | $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ | |
| How can you tell if it is horizontal? | | |
| Draw the shape here. Label a , b , and c on the drawing. | | |
| Center | (h, k) | |
| What a represents on the graph | | |
| What b represents on the graph | | |

TABLE 13.1: (continued)

| | Ellipse | Hyperbola |
|---|---------|-----------|
| What c represents on the graph | | |
| Which is the biggest, a , b , or c ? | | |
| Mathematical relationship between a , b , and c . | | |

$$2. \frac{y^2}{\frac{1}{4}} - \frac{(x-2)^2}{9} = 1$$

- Is it horizontal or vertical?
- What is the center?
- What is a ?
- What is b ?
- What is c ?
- Graph it. Make sure the box and asymptotes can be clearly seen in your graph.

$$3. 2x^2 + 8x - 4y^2 + 4y = 6$$

- Put in standard form.
 - Is it horizontal or vertical?
 - What is the center?
 - What is a ?
 - What is b ?
 - What is c ?
 - Graph it. Make sure the box and asymptotes can be clearly seen in your graph.
 - What is the equation
- A hyperbola has vertices at the origin and $(10, 0)$. One focus is at $(12, 0)$. Find the equation for the hyperbola.
 - A hyperbola has vertices at $(1, 2)$ and $(1, 22)$, and goes through the origin.
 - Find the equation for the hyperbola.
 - Find the coordinates of the two foci.

Name: _____

Sample Test: Conics 2 (Ellipses and Hyperbolas)

| | | | |
|-----------|---|---|------------------------------|
| | <u>Horizontal</u> | <u>Vertical</u> | <u>Equations</u> |
| Ellipse | $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$ | $\frac{(x-h)^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ | $a > b$ $a^2 = b^2 + c^2$ |
| | <u>Horizontal</u> | <u>Vertical</u> | <u>Equations</u> |
| Hyperbola | $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ | $\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$ | $c^2 = a^2 + b^2$ |

- Identify each equation as a line, parabola, circle, ellipse, or hyperbola.

- a. $y = \frac{x}{3}$ _____
- b. $y = \frac{3}{x}$ _____
- c. $4y^2 = 7x + 7y + 7$ _____
- d. $5(x + 3)^2 - 5(y + 3)^2 = 9$ _____
- e. $3x^2 + 3x + 6y + 3y^2 = 4x + 7$ _____
- f. $4x^2 + 5y^2 = 4$ _____

2. For each shape, is it a function or not? (Just answer yes or no.)

- a. Vertical line _____
- b. Horizontal line _____
- c. Vertical parabola _____
- d. Horizontal parabola _____
- e. Circle _____
- f. Vertical ellipse _____
- g. Horizontal ellipse _____
- h. Vertical hyperbola _____
- i. Horizontal hyperbola _____

3. The United States Capitol building contains an elliptical room. It is 96 feet in length and 46 feet in width.

- a. Write an equation to describe the shape of the room. Assume that it is centered on the origin and that the major axis is horizontal.
- b. John Quincy Adams discovered that if he stood at a certain spot in this elliptical chamber, he could overhear conversations being whispered at the opposing party leader's desk. This is because both the desk, and the secret listening spot, where foci
- c. How far was Adams standing from the edge of the room closest to him?

4. A comet zooms in from outer space, whips around the sun, and zooms back out. Its path is one branch of a hyperbola, with the sun at one of the foci. Just at the vertex, the comet is 10 million miles from the center of the hyperbola, and 15 million miles from the sun. Assume the hyperbola is horizontal, and the *center* of the hyperbola is at $(0, 0)$.

- a. Find the equation of the hyperbola.
- b. When the comet is very far away from the sun, its path is more or less a line. As you might guess, that line is represented by the asymptotes of the hyperbola. (One asymptote as it comes in, another as it goes out.) Write the equation for the line that describes the path of the comet after it has left

5. $4x^2 - 36y^2 + 144y = 153$

- a. Put in standard form.
- b. Is it horizontal or vertical?
- c. What is the center?
- d. How long is the transverse axis?
- e. How long is the conjugate axis?
- f. What are the coordinates of the two foci?
- g. Graph it. I will be looking for the vertices (the endpoints of the transverse axis), and for the asymptotes to be

drawn correctly.

Extra credit

Consider a hyperbola with foci at $(-5, 0)$ and $(5, 0)$. In order to be on the hyperbola, a point must have the following property: its distance to one focus, *minus* its distance to the other focus, must be 6.

Write the equation for this hyperbola by using the geometric definition of a hyperbola (3 points). Then simplify it to standard form (2 points).

CHAPTER **14**

Teacher's Guide

Chapter Outline

14.1 THE LONG RAMBLING PHILOSOPHICAL INTRODUCTION

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14.1 The Long Rambling Philosophical Introduction

What you're holding in your hand is much closer to a set of detailed lesson plans than to a traditional textbook. As you read through it, your first reaction may be "Who does he think he is, telling me exactly what to say and when to say it?"

Please don't take it that way. Take it this way instead.

Over a period of time, I have developed a set of in-class assignments, homeworks, and lesson plans, that work